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W -algebras for non-abelian Toda systems

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Abstract

We construct the classical W -algebras for some non-abelian Toda systems associated with the Lie groups $GL_{2n}(\mathbb{R})$ and $Sp_n(\mathbb{R})$. We start with the set of characteristic integrals and find the Poisson brackets for the corresponding Hamiltonian counterparts. The convenient block matrix representation for the Toda equations is used. The infinitesimal symmetry transformations generated by the elements of the W -algebras are presented.

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1. Introduction

The Toda systems constitute a remarkable class of two-dimensional integrable systems. According to the group-algebraic approach [1,2] such a system is specified by the choice of a Lie group G whose Lie algebra \mathfrak{g} is endowed with a \mathbb{Z} -gradation. There exist so-called higher grading [3,4] and multi-dimensional [5,6] generalizations of the Toda systems.

The ‘space–time’ for a Toda system is a two-dimensional manifold, and the ‘field space’ is the Lie group G_0 corresponding to the Lie subalgebra \mathfrak{g}_0 of \mathfrak{g} corresponding to zero value of the grading index. If the group G_0 is abelian the corresponding Toda system is said to be abelian, otherwise one has a non-abelian Toda system. There is a lot of papers devoted to abelian Toda systems, while non-abelian Toda systems are not very well studied yet. This

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is connected to the fact that until recently there was no convenient representation for such systems. It was shown in paper [7] that some class of non-abelian Toda systems can be represented in a simple block matrix form. Later it was proven that it is the case for all Toda systems associated with classical semisimple Lie groups [8]. This led to the renewal of the interest to this class of integrable systems; see, for example, papers [9–12].

In the present paper we investigate the symmetries of the simplest non-abelian Toda systems associated with finite dimensional Lie groups $GL_{2n}(\mathbb{R})$ and $Sp_n(\mathbb{R})$. Actually for the systems under consideration there are an evident symmetry resembling the symmetry of a Wess–Zumino–Novikov–Witten (WZNW) model [13,14] and the conformal symmetry. These symmetries do not exhaust all symmetries of the systems. More symmetries can be found using the so-called characteristic integrals whose existence is related to the integrability of Toda systems. These integrals give an infinite set of the densities of conserved charges. The Hamiltonian counterparts of these conserved charges generate the required symmetry transformations.

Thus, our strategy is as follows. We find the characteristic integrals for our systems (Section 3: Eqs. (3.8) and (3.11); Section 7: Eqs. (7.5) and (7.6)). Then we proceed to the Hamiltonian formalism (throughout Sections 4 and 5) and find the Hamiltonian counterparts of the characteristic integrals (Section 6: Eqs. (6.7), (6.8), (6.12) and (6.13); Section 7: Eqs. (7.9)–(7.12)) and conserved charges (Section 6: Eqs. (6.27) and (6.31)). This allows us to find the form of infinitesimal symmetry transformations in the Hamiltonian formalism and write down their Lagrangian version (Section 6: Eqs. (6.28), (6.29), (6.32) and (6.33)).

We show also that the set of characteristic integrals is closed with respect to the Poisson bracket and form an object usually called a classical W -algebra (Section 6: Eqs. (6.9)–(6.11), (6.14)–(6.16), (6.30) and (6.34)). The distinctive features of such algebras is that their defining relations are essentially non-linear and that they contain Virasoro algebras corresponding to the conformal invariance. The systematic study of W -algebras in the framework of general quantum conformal field theory was initiated by Zamolodchikov [15]. For a detailed review of the subject we refer the reader to paper [16].

Although our paper contains original results, in some parts it has character of a review. It is worth to note here that majority of the results on W -algebras for Toda systems was obtained by the method of Hamiltonian reduction that is based on the fact that Toda systems can be obtained if one starts with a WZNW model based on a Lie group G and then imposes relevant constraints on the conserved currents forming with respect to the Poisson bracket two copies of loop algebras associated with the Lie algebra \mathfrak{g} [17–19]. Here the Toda ‘field space’ arises as a factor in the generalized Gauss decomposition of the Lie group G , which is valid only for a dense subset of G . This results in that the true reduced system is different from a Toda system; see, in this respect, papers [20–26]. Such our conclusion is justified at least by the fact that the Toda systems have singular solutions corresponding to some non-singular initial conditions, and that is impossible for a system being a reduction of a WZNW model which does not have such solutions. Thus, the results on Toda systems obtained with the help of the method of Hamiltonian reduction require verification.

It seems to us that the direct method used in our paper is more appropriate to the problem under consideration than the method of Hamiltonian reduction. In particular, it allows to identify the generators of the Virasoro algebras describing the conformal properties of the model with the Hamiltonian counterparts of the components of the conformally improved

energy–momentum tensor which is constructed by a standard procedure (see Section 6: Eqs. (6.23)–(6.26); Section 7: Eqs. (7.13)–(7.16)).

2. Toda systems

In accordance with the group-algebraic approach [1,2] the construction of equations describing a Toda system looks as follows. Let G be a real or complex Lie group whose Lie algebra \mathfrak{g} is endowed with a \mathbb{Z} -gradation,

$$\mathfrak{g} = \bigoplus_{m \in \mathbb{Z}} \mathfrak{g}_m, \quad [\mathfrak{g}_m, \mathfrak{g}_n] \subset \mathfrak{g}_{m+n}.$$

Recall that for a given \mathbb{Z} -gradation of \mathfrak{g} the subspace \mathfrak{g}_0 is a subalgebra of \mathfrak{g} . The subspaces

$$\mathfrak{g}_{<0} = \bigoplus_{m < 0} \mathfrak{g}_m, \quad \mathfrak{g}_{>0} = \bigoplus_{m > 0} \mathfrak{g}_m$$

are also subalgebras of \mathfrak{g} . Denote by G_0 , $G_{<0}$ and $G_{>0}$ the connected Lie subgroups of G corresponding to the subalgebras \mathfrak{g}_0 , and $\mathfrak{g}_{<0}$ and $\mathfrak{g}_{>0}$, respectively.

Let M be a real two-dimensional manifold. Introduce local coordinates on M and denote them by z^- and z^+ . One can also consider the case when M is a one-dimensional complex manifold. In this case z^- is a complex coordinate on M and z^+ is the complex conjugate of z^- . Let a_- and a_+ be some fixed mappings from M to \mathfrak{g}_{-1} and \mathfrak{g}_{+1} , respectively, satisfying the relations

$$\partial_+ a_- = 0, \quad \partial_- a_+ = 0. \tag{2.1}$$

Here and below we denote the partial derivatives over z^- and z^+ by ∂_- and ∂_+ . Actually we assume that the subspaces \mathfrak{g}_{-1} and \mathfrak{g}_{+1} are non-trivial. Generally, if l is a positive integer such that the subspaces \mathfrak{g}_m are trivial for $-l < m < 0$ and $0 < m < l$, one defines a_- and a_+ as mappings from M to \mathfrak{g}_{-l} and \mathfrak{g}_{+l} , respectively. Restrict ourselves to the case when G is a matrix Lie group. In other words, assume that for some positive integer N it is a Lie subgroup of the Lie group $GL_N(\mathbb{R})$ or of the Lie group $GL_N(\mathbb{C})$. More general case is discussed in paper [27]. In the case under consideration the equations describing the Toda system are matrix partial differential equations of the form

$$\partial_+(\gamma^{-1} \partial_- \gamma) = [a_-, \gamma^{-1} a_+ \gamma], \tag{2.2}$$

where γ is a mapping from M to G_0 . Note that Eq. (2.2) can also be written as

$$\partial_-(\partial_+ \gamma \gamma^{-1}) = [\gamma a_- \gamma^{-1}, a_+]. \tag{2.3}$$

Parametrizing the group G_0 by a set of independent parameters, or, in other words, introducing some coordinates on G_0 , we can rewrite the Toda equations as a system of equations for ordinary functions, which we call Toda fields.

If the Lie group G_0 is abelian we say that we deal with an abelian Toda system, otherwise we call the system a non-abelian one. The complete classification of the Toda systems associated with the classical Lie groups is given in paper [8].

There is a constructive procedure of obtaining the general solution to Toda equations [2,5,6]. It is based on the use of the Gauss decomposition related to the \mathbb{Z} -gradation under consideration. Here the Gauss decomposition is the representation of an element of the Lie group G as a product of elements of the subgroups $G_{<0}$, $G_{>0}$ and G_0 taken in an appropriate order. Another approach is based on the theory of representations of Lie groups [1,2].

In this paper we consider the simplest examples of non-abelian Toda equations based on the Lie groups $GL_{2n}(\mathbb{R})$ and $Sp_n(\mathbb{R})$ [6,28].

We start with the Lie group $G = GL_{2n}(\mathbb{R})$. The case of the Lie group $Sp_n(\mathbb{R})$ will be considered in Section 7. The Lie algebra $\mathfrak{g} = \mathfrak{gl}_{2n}(\mathbb{R})$ of $GL_{2n}(\mathbb{R})$ is formed by all real $2n \times 2n$ matrices. Below we represent an arbitrary $2n \times 2n$ matrix x in the block matrix form

$$x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \tag{2.4}$$

where x_{rs} , $r, s = 1, 2$, are $n \times n$ matrices.

Recall that an element $q \in \mathfrak{g}$ is said to be the grading operator generating the \mathbb{Z} -gradation under consideration if

$$\mathfrak{g}_m = \{x \in \mathfrak{g} | [q, x] = mx\}.$$

In particular, any \mathbb{Z} -gradation of a finite dimensional complex semisimple Lie algebra is generated by the corresponding grading operator.

Denote by I_n the unit $n \times n$ matrix. It is easy to show that the element

$$q = \frac{1}{2} \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} \tag{2.5}$$

generates a \mathbb{Z} -gradation of $\mathfrak{gl}_{2n}(\mathbb{R})$. Here the subspaces \mathfrak{g}_{-1} and \mathfrak{g}_{+1} are the sets formed by all block strictly lower triangular and strictly upper triangular matrices of $\mathfrak{gl}_{2n}(\mathbb{R})$, respectively, and the subspace \mathfrak{g}_0 is the set of all block diagonal matrices of $\mathfrak{gl}_{2n}(\mathbb{R})$. All other grading subspaces are trivial, and we have $\mathfrak{g}_{<0} = \mathfrak{g}_{-1}$, $\mathfrak{g}_{>0} = \mathfrak{g}_{+1}$.

Hence, the general form of the mappings a_- and a_+ is

$$a_- = \begin{pmatrix} 0 & 0 \\ A_- & 0 \end{pmatrix}, \quad a_+ = \begin{pmatrix} 0 & A_+ \\ 0 & 0 \end{pmatrix}, \tag{2.6}$$

where A_- and A_+ are arbitrary $n \times n$ matrix-valued functions on M satisfying the condition

$$\partial_+ A_- = 0, \quad \partial_- A_+ = 0. \tag{2.7}$$

In this paper we restrict ourselves to the case $A_- = I_n$ and $A_+ = I_n$.

It is not difficult to describe the corresponding subgroups $G_{<0}$, $G_{>0}$ and G_0 of the Lie group $GL_{2n}(\mathbb{R})$. The subgroups $G_{<0}$ and $G_{>0}$ consist of all block lower triangular and upper triangular matrices of $GL_{2n}(\mathbb{R})$, respectively, with unit matrices on the diagonal. The subgroup G_0 is formed by all block diagonal matrices of $GL_{2n}(\mathbb{R})$.

Parameterize the mapping γ as

$$\gamma = \begin{pmatrix} \Gamma^{(1)} & 0 \\ 0 & \Gamma^{(2)} \end{pmatrix}, \tag{2.8}$$

where $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are mappings from M to the Lie group $GL_n(\mathbb{R})$. With this parameterization we write the Toda equations in the form

$$\partial_+(\Gamma^{(1)-1}\partial_-\Gamma^{(1)}) = -\Gamma^{(1)-1}\Gamma^{(2)}, \quad \partial_+(\Gamma^{(2)-1}\partial_-\Gamma^{(2)}) = \Gamma^{(1)-1}\Gamma^{(2)}, \quad (2.9)$$

or in the form

$$\partial_-(\partial_+\Gamma^{(1)}\Gamma^{(1)-1}) = -\Gamma^{(2)}\Gamma^{(1)-1}, \quad \partial_-(\partial_+\Gamma^{(2)}\Gamma^{(2)-1}) = \Gamma^{(2)}\Gamma^{(1)-1}. \quad (2.10)$$

The exact general solution to these equations was obtained in paper [6].

One can get convinced that the transformations

$$\Gamma^{(1)} \rightarrow \Lambda_+\Gamma^{(1)}\Lambda_-, \quad \Gamma^{(2)} \rightarrow \Lambda_+\Gamma^{(2)}\Lambda_-, \quad (2.11)$$

where Λ_- and Λ_+ are mappings from M to $GL_n(\mathbb{R})$ satisfying the conditions

$$\partial_+\Lambda_- = 0, \quad \partial_-\Lambda_+ = 0$$

are symmetry transformations for the system under consideration. This symmetry, by an evident reason, can be called a WZNW-type symmetry.

The system possesses also the conformal symmetry. Here the conformal transformations

$$z^- \rightarrow \zeta^-(z^-), \quad z^+ \rightarrow \zeta^+(z^+) \quad (2.12)$$

act on the space of solutions of Eqs. (2.9), or (2.10), in the following way [8]:

$$\Gamma^{(1)}(z^-, z^+) \rightarrow [\partial_-\zeta^-(z^-)\partial_+\zeta^+(z^+)]^{-1/2}\Gamma^{(1)}(\zeta^+(z^+), \zeta^-(z^-)), \quad (2.13)$$

$$\Gamma^{(2)}(z^-, z^+) \rightarrow [\partial_-\zeta^-(z^-)\partial_+\zeta^+(z^+)]^{1/2}\Gamma^{(2)}(\zeta^+(z^+), \zeta^-(z^-)). \quad (2.14)$$

The WZNW-type symmetry and the conformal symmetry do not exhaust all symmetries of the system. To find additional symmetry transformations we can use the following procedure.

First we find conserved charges. In the case under consideration we have an infinite set of conserved charges provided by the so-called characteristic integrals. In the Hamiltonian formalism the conserved charges generate symmetry transformations. So, we construct the Lagrangian formulation for our system and then proceed to the corresponding Hamiltonian description. After that we consider the symmetry transformations generated by the Hamiltonian counterparts of the conserved charges associated with the characteristic integrals, and finally obtain their Lagrangian version. This allows us, in particular, to obtain the WZNW-type symmetry transformations and the conformal transformations discussed above.

3. Characteristic integrals

A characteristic integral of a Toda system is, by definition, either a differential polynomial W of the Toda fields satisfying the relation

$$\partial_+W = 0, \quad (3.1)$$

or a differential polynomial \bar{W} of the Toda fields which satisfy the relation

$$\partial_- \bar{W} = 0. \tag{3.2}$$

By a differential polynomial we mean a polynomial function of the fields and their derivatives.

Let us treat the manifold M as a flat Riemannian manifold with the coordinates z^- and z^+ being light-front coordinates and the metric tensor η having the form (A.2). The usual flat coordinates z^0 and z^1 are related to the light-front coordinates z^- and z^+ by the relation (A.1). Using these coordinates we write the equality (3.1) as

$$\partial_0 W + \partial_1 W = 0,$$

where $\partial_0 = \partial/\partial z^0$ and $\partial_1 = \partial/\partial z^1$. Hence, the function W is a density of a conserved charge. Moreover, multiplying W by a function which depends only on z^- we again obtain a characteristic integral. Therefore, a characteristic integral generates an infinite set of densities of conserved charges. Similarly, multiplying a characteristic integral satisfying the relation (3.2) by functions depending only on z^+ we again obtain an infinite set of densities of conserved charges.

It is clear that any differential polynomial of characteristic integrals is also a characteristic integral. Moreover, the Poisson bracket of the Hamiltonian counterparts of any two characteristic integrals is again a characteristic integral. Therefore, a necessary step in investigation of characteristic integrals is to show that they form a closed set with respect to the Poisson bracket, or, in other words, that they form an object called a W -algebra; see, for a review, [16].

There are two main methods for obtaining characteristic integrals for Toda systems. The first one is based on the construction of a generating pseudo-differential operator; see, for example, papers [17,29–31]. The second method is based on the usage of the so-called Drinfeld–Sokolov gauge; see, for example, papers [17–19,32]. In the present paper we use the latter method.

It is well known that Toda equation (2.2) can be obtained as the zero curvature condition for some connection on the trivial principal fiber bundle $M \times G \rightarrow M$ [1,2]. We identify the connection under consideration with a \mathfrak{g} -valued one-form ω on M . Using the basis formed by the 1-forms dz^- and dz^+ , we write

$$\omega = \omega_- dz^- + \omega_+ dz^+,$$

where the components ω_- and ω_+ are \mathfrak{g} -valued functions on M . The curvature of the connection ω is zero if and only if

$$d\omega + \omega \wedge \omega = 0, \tag{3.3}$$

or, in terms of the components,

$$\partial_- \omega_+ - \partial_+ \omega_- + [\omega_-, \omega_+] = 0. \tag{3.4}$$

If we consider the components ω_- and ω_+ of the form

$$\omega_- = a_- + \gamma^{-1} \partial_- \gamma, \quad \omega_+ = \gamma^{-1} a_+ \gamma, \tag{3.5}$$

then the zero curvature condition (3.4) is equivalent to the Toda equation (2.2).

Recall that the zero curvature condition is gauge invariant. It means that if a connection ω satisfies the relation (3.3), then for any mapping $\psi : M \rightarrow G$ the gauge transformed connection

$$\omega^\psi = \psi^{-1}\omega\psi + \psi^{-1}d\psi$$

satisfies the relation (3.3) as well. In terms of the components one has

$$\omega_-^\psi = \psi^{-1}\omega_-\psi + \psi^{-1}\partial_-\psi, \quad \omega_+^\psi = \psi^{-1}\omega_+\psi + \psi^{-1}\partial_+\psi.$$

In particular, if we consider the connection with the components given by (3.5) and choose $\psi = \gamma^{-1}$ we will come to the connection, which we also denote by ω , with the components

$$\omega_- = \gamma a_- \gamma^{-1}, \quad \omega_+ = -\partial_+ \gamma \gamma^{-1} + a_+. \tag{3.6}$$

And the zero curvature condition (3.4) gives the Toda equations written in form (2.3).

Let us return to our specific example of Toda equations. Write the components ω_- and ω_+ defined by (3.5) in the block matrix form

$$\omega_- = \begin{pmatrix} \Sigma_-^{(1)} & 0 \\ I_n & \Sigma_-^{(2)} \end{pmatrix}, \quad \omega_+ = \begin{pmatrix} 0 & \Gamma^{(1)-1}\Gamma^{(2)} \\ 0 & 0 \end{pmatrix},$$

where we denoted

$$\Sigma_-^{(1)} = \Gamma^{(1)-1}\partial_-\Gamma^{(1)}, \quad \Sigma_-^{(2)} = \Gamma^{(2)-1}\partial_-\Gamma^{(2)}. \tag{3.7}$$

Now consider the gauge transformation generated by a mapping $\psi : M \rightarrow G_{>0}$. The general form of such a mapping is

$$\psi = \begin{pmatrix} I_n & \chi \\ 0 & I_n \end{pmatrix}.$$

For the component ω_-^ψ we obtain the expression

$$\omega_-^\psi = \begin{pmatrix} \Sigma_-^{(1)} - \chi & (\Sigma_-^{(1)} - \chi)\chi - \chi\Sigma_-^{(2)} + \partial_-\chi \\ I_n & \Sigma_-^{(2)} + \chi \end{pmatrix}.$$

The Drinfeld–Sokolov gauge [17–19,32] in our case is fixed by the requirement

$$(\omega_-^\psi)_{11} = (\omega_-^\psi)_{22}.$$

It is clear that this requirement gives

$$\chi = \frac{1}{2}(\Sigma_-^{(1)} - \Sigma_-^{(2)}),$$

and we obtain

$$\omega_-^\psi = \begin{pmatrix} -\frac{1}{2\kappa}W_1 & -\frac{1}{\kappa^2}W_2 + \frac{1}{4\kappa^2}W_1^2 \\ I_n & -\frac{1}{2\kappa}W_1 \end{pmatrix},$$

where

$$W_1 = -\kappa(\Sigma_-^{(1)} + \Sigma_-^{(2)}), \quad W_2 = -\kappa^2(\frac{1}{2}(\partial_- \Sigma_-^{(1)} - \partial_- \Sigma_-^{(2)}) - \Sigma_-^{(1)} \Sigma_-^{(2)}), \quad (3.8)$$

and κ is a constant. We introduced the constant κ in the definition of the quantities W_1 and W_2 for future convenience. Actually we will identify it with the constant entering the action of the Toda theory.

For the component ω_+^ψ we have the expression

$$\omega_+^\psi = \begin{pmatrix} 0 & \Gamma^{(1)-1} \Gamma^{(2)} + \frac{1}{2}(\partial_+ \Sigma_-^{(1)} - \partial_+ \Sigma_-^{(2)}) \\ 0 & 0 \end{pmatrix}.$$

Therefore, if $\Gamma^{(1)}$ and $\Gamma^{(2)}$ satisfy the Toda equations (2.9) then $\omega_+^\psi = 0$, and the zero curvature condition gives

$$\partial_+ \omega_-^\psi = 0.$$

This equality implies that

$$\partial_+ W_1 = 0, \quad \partial_+ W_2 = 0. \quad (3.9)$$

Thus, the quantities W_1 and W_2 are matrix characteristic integrals of the Toda system under consideration.

As we have already noted, any differential polynomial of characteristic integrals is a characteristic integral. Therefore, they form a differential algebra. The generators of this algebra, the matrix elements of W_1 and W_2 in our case, can be chosen in different ways. Our choice is inspired by an intention to get simple expressions for Poisson brackets.

One can also start with the connection components of the form (3.6). Performing the gauge transformation with $\psi : M \rightarrow G_{<0}$,

$$\psi = \begin{pmatrix} I_n & 0 \\ \frac{1}{2}(\bar{\Sigma}_+^{(1)} - \bar{\Sigma}_+^{(2)}) & I_n \end{pmatrix},$$

where

$$\bar{\Sigma}_+^{(1)} = \partial_+ \Gamma^{(1)} \Gamma^{(1)-1}, \quad \bar{\Sigma}_+^{(2)} = \partial_+ \Gamma^{(2)} \Gamma^{(2)-1}, \quad (3.10)$$

we obtain

$$\omega_+^\psi = \begin{pmatrix} \frac{1}{2\kappa} \bar{W}_1 & I_n \\ -\frac{1}{\kappa^2} \bar{W}_2 + \frac{1}{4\kappa^2} \bar{W}_1^2 & \frac{1}{2\kappa} \bar{W}_1 \end{pmatrix},$$

where

$$\bar{W}_1 = -\kappa(\bar{\Sigma}_+^{(1)} + \bar{\Sigma}_+^{(2)}), \quad \bar{W}_2 = -\kappa^2(\frac{1}{2}(\partial_+ \bar{\Sigma}_+^{(1)} - \partial_+ \bar{\Sigma}_+^{(2)}) - \bar{\Sigma}_+^{(2)} \bar{\Sigma}_+^{(1)}). \quad (3.11)$$

For the component ω_-^ψ one has

$$\omega_-^\psi = \begin{pmatrix} 0 & 0 \\ \Gamma^{(2)}\Gamma^{(1)-1} + \frac{1}{2}(\partial_- \bar{\Sigma}_+^{(1)} - \partial_- \bar{\Sigma}_+^{(2)}) & 0 \end{pmatrix},$$

and the Toda equations (2.10) give $\omega_-^\psi = 0$. The zero curvature condition implies that

$$\partial_- \bar{W}_1 = 0, \quad \partial_- \bar{W}_2 = 0, \tag{3.12}$$

and we end up with another set of characteristic integrals.

4. Lagrangian formalism for Toda systems

To write the action describing a Toda system we must be able to integrate over the manifold M , in other words, we have to define a volume form. To this end, as in the previous section, we treat the manifold M as a flat Riemannian manifold with a metric tensor η . The next ingredient needed is a non-degenerate symmetric invariant scalar product in the Lie algebra \mathfrak{g} . We assume that \mathfrak{g} is endowed with such a scalar product and denote it by B .

The action functional $S[\gamma]$ of a Toda system is the sum of three terms:

$$S[\gamma] = S_C[\gamma] + S_{WZ}[\gamma] + S_T[\gamma].$$

Let us discuss them in order.

The first term $S_C[\gamma]$ is the action functional of the principal chiral field model. Using some arbitrary coordinates on M , denoted by z^i , we write

$$S_C[\gamma] = -\frac{\kappa}{2} \int_M \eta^{ij} B(\gamma^{-1} \partial_i \gamma, \gamma^{-1} \partial_j \gamma) \sqrt{|\eta|} d^2 z,$$

where κ is a constant. Note that if γ is a mapping from M to G_0 , then $\gamma^{-1} \partial_i \gamma$ is a mapping from M to \mathfrak{g}_0 .

The second term is the so-called Wess–Zumino term which is constructed as follows. Suppose that the manifold M is the boundary of the three-dimensional manifold \tilde{M} , $M = \partial \tilde{M}$. Let $\tilde{\gamma}$ be an extension of the mapping γ from M to \tilde{M} . The Wess–Zumino term is

$$S_{WZ}[\gamma] = -\frac{\kappa}{3!} \int_{\tilde{M}} \epsilon^{JK} B(\tilde{\gamma}^{-1} \partial_I \tilde{\gamma}, [\tilde{\gamma}^{-1} \partial_J \tilde{\gamma}, \tilde{\gamma}^{-1} \partial_K \tilde{\gamma}]) d^3 \tilde{z},$$

where \tilde{z}^I are some coordinates on \tilde{M} and ϵ^{JK} is the absolutely skew-symmetric symbol. It can be shown that the variations of the Wess–Zumino term are determined by the mapping γ only. Hence the corresponding equations of motion govern the mapping γ , leaving the extension $\tilde{\gamma}$ arbitrary. It is an example of the so-called multi-valued functional, and so, we write just $S_{WZ}[\gamma]$ instead of $S_{WZ}[\tilde{\gamma}]$.

The last term is the Toda term, which has the form

$$S_T[\gamma] = \kappa \int_M B(a_-, \gamma^{-1} a_+ \gamma) \sqrt{|\eta|} d^2 z.$$

Here a_- and a_+ are fixed mappings from M to \mathfrak{g}_{-1} and \mathfrak{g}_{+1} , respectively, satisfying the conditions

$$(\sqrt{|\eta|}\eta^{ij} + \epsilon^{ij})\partial_j a_+ = 0, \quad (\sqrt{|\eta|}\eta^{ij} - \epsilon^{ij})\partial_j a_- = 0. \tag{4.1}$$

The action functional of the WZNW model is the sum of the functionals $S_C[\gamma]$ and $S_{WZ}[\gamma]$. The functional S_T does not contain derivatives of γ . Therefore, the construction of the Hamiltonian formalism for a Toda system is a trivial modification of that for the WZNW model.

Let us show that the action $S[\gamma]$ does really give the Toda equations. One finds consecutively

$$\delta S_C[\gamma] = \kappa \int_M B \left(\gamma^{-1} \delta \gamma, \frac{1}{\sqrt{|\eta|}} \partial_i (\sqrt{|\eta|} \eta^{ij} \gamma^{-1} \partial_j \gamma) \right) \sqrt{|\eta|} d^2 z, \tag{4.2}$$

$$\delta S_{WZ}[\gamma] = \kappa \int_M B \left(\gamma^{-1} \delta \gamma, \frac{1}{\sqrt{|\eta|}} \epsilon^{ij} \partial_i (\gamma^{-1} \partial_j \gamma) \right) \sqrt{|\eta|} d^2 z, \tag{4.3}$$

$$\delta S_T[\gamma] = \kappa \int_M B(\gamma^{-1} \delta \gamma, [a_-, \gamma^{-1} a_+ \gamma]) \sqrt{|\eta|} d^2 z. \tag{4.4}$$

To obtain from these relations the equations of motion one should use the fact that the restriction of the scalar product B to the Lie subalgebra \mathfrak{g}_0 is non-degenerate. To show this let us take two elements, x_m and x_n , belonging to \mathfrak{g}_m and \mathfrak{g}_n respectively. From (B.18) it follows that

$$B([x_m, q], x_n) = B(x_m, [q, x_n]),$$

and one obtains

$$(m + n)B(x_m, x_n) = 0.$$

Therefore, $B(x_m, x_n) = 0$ if $n + m \neq 0$. This implies that the restriction of the scalar product B to \mathfrak{g}_0 is non-degenerate indeed. Note also that

$$B|_{\mathfrak{g}_{<0}} = 0, \quad B|_{\mathfrak{g}_{>0}} = 0,$$

and that B gives a non-degenerate pairing of the nilpotent subalgebras $\mathfrak{g}_{<0}$ and $\mathfrak{g}_{>0}$.

Since the scalar product $B|_{\mathfrak{g}_0}$ is non-degenerate, the relations (4.2)–(4.4) give rise to the following equations of motion

$$\frac{1}{\sqrt{|\eta|}} \partial_i (\sqrt{|\eta|} \eta^{ij} \gamma^{-1} \partial_j \gamma + \epsilon^{ij} \gamma^{-1} \partial_j \gamma) + [a_-, \gamma^{-1} a_+ \gamma] = 0. \tag{4.5}$$

Using light-front coordinates, one sees that these equations coincide with the Toda equation (2.2). Here the conditions (4.1) coincide with the conditions (2.1). Rewriting Eq. (4.5) as

$$\frac{1}{\sqrt{|\eta|}} \partial_i (\sqrt{|\eta|} \eta^{ij} \partial_j \gamma \gamma^{-1} - \epsilon^{ij} \partial_j \gamma \gamma^{-1}) + [\gamma a_- \gamma^{-1}, a_+] = 0,$$

and using light-front coordinates, we come to Eq. (2.3).

It is convenient now to introduce some coordinates y^μ in G_0 and work in terms of fields ξ^μ defined as

$$\xi^\mu = y^\mu \circ \gamma = \gamma^* y^\mu.$$

Let g be the matrix-valued function which transforms the coordinates $y^\mu(a)$ of the element $a \in G_0$ into the element a itself, then we can write

$$\gamma = g(\xi).$$

Therefore, one has

$$\gamma^{-1} \partial_i \gamma = e_\alpha \theta_\mu^\alpha(\xi) \partial_i \xi^\mu,$$

where $\{e_\alpha\}$ is a basis of \mathfrak{g}_0 , and the functions θ_μ^α are defined in [Appendix B](#). Using this relation, we obtain for the density of the Lagrangian of the principal chiral field model the expression

$$\mathcal{L}_C = -\frac{1}{2} \kappa \sqrt{|\eta|} c_{\alpha\beta} \theta_\mu^\alpha(\xi) \theta_\nu^\beta(\xi) \eta^{ij} \partial_i \xi^\mu \partial_j \xi^\nu,$$

where the quantities $c_{\alpha\beta}$ are given by [\(B.19\)](#). Introducing the notation

$$h_{\mu\nu}(y) = c_{\alpha\beta} \theta_\mu^\alpha(y) \theta_\nu^\beta(y), \tag{4.6}$$

we write the density of the Lagrangian \mathcal{L}_C as

$$\mathcal{L}_C = -\frac{1}{2} \kappa \sqrt{|\eta|} h_{\mu\nu}(\xi) \eta^{ij} \partial_i \xi^\mu \partial_j \xi^\nu.$$

Note that $h_{\mu\nu}(y)$ are the components of the bi-invariant metric tensor on the Lie group G_0 .

The Wess–Zumino term can be written as

$$S_{WZ}[\gamma] = -\kappa \int_{\tilde{M}} \tilde{\gamma}^* \Theta,$$

where the three-form Θ is given by the relation [\(B.21\)](#). Using the local representation [\(B.22\)](#), we obtain

$$S_{WZ}[\gamma] = -\kappa \int_M \gamma^* \lambda$$

that gives

$$\mathcal{L}_{WZ} = -\frac{1}{2} \kappa \lambda_{\mu\nu}(\xi) \epsilon^{ij} \partial_i \xi^\mu \partial_j \xi^\nu.$$

Finally, for the contribution to the density of the Lagrangian of the Toda system, which is due to the term $S_T[\gamma]$, we have

$$\mathcal{L}_T = \kappa \sqrt{|\eta|} B(a_-, g^{-1}(\xi) a_+ g(\xi)) = -\kappa \sqrt{|\eta|} V(\xi).$$

Collecting all terms, we come to the following expression for the density of the Lagrangian of a Toda system

$$\mathcal{L} = -\frac{1}{2} \kappa \sqrt{|\eta|} h_{\mu\nu}(\xi) \eta^{ij} \partial_i \xi^\mu \partial_j \xi^\nu - \frac{1}{2} \kappa \lambda_{\mu\nu}(\xi) \epsilon^{ij} \partial_i \xi^\mu \partial_j \xi^\nu - \kappa \sqrt{|\eta|} V(\xi). \tag{4.7}$$

Let us restrict ourselves to the case when the mappings a_- and a_+ are constant. In this case the Toda system under consideration is conformally invariant, and it is possible to define

the energy–momentum tensor for it being symmetric and traceless. Recall that there are two standard methods to define the energy–momentum tensor. The first method is the variation of the action over the components of the metric tensor that gives the so-called symmetric energy–momentum tensor. Note that in our case the Wess–Zumino term does not depend on the metric, therefore it does not give a contribution to the symmetric energy–momentum tensor. It may seem to be strange at the first glance. To demonstrate that this is, however, the case, we start with the canonical energy–momentum tensor T^i_j which is defined by

$$\sqrt{|\eta|}T^i_j = -\frac{\partial \mathcal{L}}{\partial(\partial_i \xi^\mu)} \partial_j \xi^\mu + \delta^i_j \mathcal{L}.$$

It is convenient to write the expression for the components of the energy–momentum tensor with upper indices. We have

$$\begin{aligned} \sqrt{|\eta|}T^{ij} &= \kappa\sqrt{|\eta|}h_{\mu\nu}(\xi)\eta^{ik}\eta^{jl}\partial_k \xi^\mu \partial_l \xi^\nu - \kappa\lambda_{,\mu\nu}(\xi)\epsilon^{ik}\eta^{jl}\partial_k \xi^\mu \partial_l \xi^\nu \\ &+ \eta^{ij}\left(-\frac{1}{2}\kappa\sqrt{|\eta|}h_{\mu\nu}(\xi)\eta^{kl}\partial_k \xi^\mu \partial_l \xi^\nu - \frac{1}{2}\kappa\lambda_{,\mu\nu}(\xi)\epsilon^{kl}\partial_k \xi^\mu \partial_l \xi^\nu - \kappa\sqrt{|\eta|}V(\xi)\right). \end{aligned}$$

Consider the terms containing $\lambda_{,\mu\nu}(\xi)$. They can be written as

$$-\frac{1}{2}\kappa(\epsilon^{ik}\eta^{lj} + \epsilon^{li}\eta^{kj} + \epsilon^{kl}\eta^{ij})\lambda_{,\mu\nu}(\xi)\partial_k \xi^\mu \partial_l \xi^\nu.$$

The sum $\epsilon^{ik}\eta^{lj} + \epsilon^{li}\eta^{kj} + \epsilon^{kl}\eta^{ij}$ is totally antisymmetric with respect to the indices i, k and l . Since we work in a two-dimensional space–time, this sum is equal to zero, and we can write

$$T^{ij} = \kappa\eta^{ik}\eta^{jl}h_{\mu\nu}(\xi)\partial_k \xi^\mu \partial_l \xi^\nu - \frac{1}{2}\kappa\eta^{ij}\eta^{kl}h_{\mu\nu}(\xi)\partial_k \xi^\mu \partial_l \xi^\nu - \kappa\eta^{\mu\nu}V(\xi).$$

Thus, the canonical energy–momentum tensor of a Toda system has no terms arising from the Wess–Zumino term. It can be shown that it coincides with the symmetric energy–momentum tensor. For the symmetric energy–momentum tensor one has

$$T^i_{;i} = 0,$$

where the usual notation for the covariant derivatives with respect to the metric tensor η is used. In terms of the mapping γ we obtain

$$\begin{aligned} T^{ij} &= \kappa\eta^{ik}\eta^{jl}B(\gamma^{-1}\partial_k \gamma, \gamma^{-1}\partial_l \gamma) - \frac{1}{2}\kappa\eta^{ij}\eta^{kl}B(\gamma^{-1}\partial_k \gamma, \gamma^{-1}\partial_l \gamma) \\ &+ \kappa\eta^{ij}B(a_-, \gamma^{-1}a_+ \gamma). \end{aligned}$$

The trace of the obtained energy–momentum tensor is different from zero, namely

$$T^i_i = 2\kappa B(a_-, \gamma^{-1}a_+ \gamma).$$

Let us construct the so-called conformally improved traceless energy–momentum tensor. To this end first note that since the mapping a_- takes values in \mathfrak{g}_{-1} , one can write

$$B(a_-, \gamma^{-1}a_+ \gamma) = -B([q, a_-], \gamma^{-1}a_+ \gamma) = -B(q, [a_-, \gamma^{-1}a_+ \gamma]).$$

Taking into account the equations of motion (4.5), we see that

$$B(a_-, \gamma^{-1} a_+ \gamma) = \frac{1}{\sqrt{|\eta|}} B(q, \partial_i (\sqrt{|\eta|} \eta^{ij} \gamma^{-1} \partial_j \gamma + \epsilon^{ij} \gamma^{-1} \partial_j \gamma)),$$

with account of the equality

$$[q, \gamma] = 0$$

can be written as

$$B(a_-, \gamma^{-1} a_+ \gamma) = \frac{1}{\sqrt{|\eta|}} \partial_i B(q, \sqrt{|\eta|} \eta^{ij} \gamma^{-1} \partial_j \gamma),$$

Thus the trace of the energy–momentum tensor can be represented in the form

$$T_i^i = 2R_{;i}^i,$$

where

$$R^i = \kappa B(q, \eta^{ij} \gamma^{-1} \partial_j \gamma) = \kappa B(q, \eta^{ij} \partial_j \gamma \gamma^{-1}).$$

Now let us use the well-known fact that the energy–momentum tensor is defined ambiguously. In particular, one can use instead of the tensor T^{ij} the tensor

$$T'^{ij} = T^{ij} + S_{;k}^{ikj},$$

where the components S^{ikj} satisfy the relation

$$S^{ikj} = -S^{kij}.$$

It is clear that one has

$$T'_{;i}{}^{ij} = 0. \tag{4.8}$$

One can easily check that with the choice

$$S^{ikj} = -2\eta^{ij} R^k + 2\eta^{kj} R^i$$

we obtain a traceless and symmetric tensor T'^{ij} . This is the conformally improved energy–momentum tensor for the Toda system.

Using coordinates for which the components of the metric tensor are constant, we come to the expression

$$\begin{aligned} T'_{ij} = & \kappa B(\gamma^{-1} \partial_i \gamma, \gamma^{-1} \partial_j \gamma) - \frac{1}{2} \kappa \eta_{ij} \eta^{kl} B(\gamma^{-1} \partial_k \gamma, \gamma^{-1} \partial_l \gamma) \\ & + 2\kappa B(q, \partial_i (\gamma^{-1} \partial_j \gamma)) - \kappa \eta_{ij} \eta^{kl} B(q, \partial_k (\gamma^{-1} \partial_l \gamma)). \end{aligned}$$

Since, the natural coordinates for a two-dimensional conformally invariant system are light-front coordinates let us write the components of the conformally improved energy–momentum tensor using such coordinates. First of all recall that since the conformally

improved energy–momentum tensor is symmetric and traceless then

$$T'_{-+} = 0, \quad T'_{+-} = 0.$$

Therefore, the relations (4.8) take the form

$$\partial_+ T'_{--} = 0, \quad \partial_- T'_{++} = 0. \tag{4.9}$$

It is convenient to choose the following explicit expressions for the non-zero components:

$$T'_{--} = \kappa B(\gamma^{-1} \partial_- \gamma, \gamma^{-1} \partial_- \gamma) + 2\kappa B(q, \partial_-(\gamma^{-1} \partial_- \gamma)), \tag{4.10}$$

$$T'_{++} = \kappa B(\partial_+ \gamma \gamma^{-1}, \partial_+ \gamma \gamma^{-1}) + 2\kappa B(q, \partial_+(\partial_+ \gamma \gamma^{-1})). \tag{4.11}$$

For the Toda system discussed in Section 2 we define the scalar product B by the relation

$$B(x, y) = \text{tr}(xy). \tag{4.12}$$

It is clear that this scalar product is symmetric, non-degenerate and Ad-invariant. Taking into account the relations (2.5), (3.7) and (3.10), we obtain

$$T'_{--} = \kappa \text{tr}[\Sigma_-^{(1)2} + \Sigma_-^{(2)2} + \partial_-(\Sigma_-^{(1)} - \Sigma_-^{(2)})],$$

$$T'_{++} = \kappa \text{tr}[\bar{\Sigma}_+^{(1)2} + \bar{\Sigma}_+^{(2)2} + \partial_+(\bar{\Sigma}_+^{(1)} - \bar{\Sigma}_+^{(2)})].$$

The definitions (3.8) and (3.11) allow us to write

$$T'_{--} = \frac{1}{\kappa} \text{tr}[W_1^2 - 2W_2], \quad T'_{++} = \frac{1}{\kappa} \text{tr}[\bar{W}_1^2 - 2\bar{W}_2]. \tag{4.13}$$

Here the equalities (4.9) can be considered as consequences of the relations (3.9) and (3.12).

5. Hamiltonian formalism

In this section we follow mainly the paper by Bowcock [33] where the Hamiltonian formulation of the WZNW model and its gauged version was investigated. The approach used in [33] is based on usage of a local representation of the closed three-form entering the definition of the action, as an exact form. Actually we used this trick in the previous section to construct the density of the Lagrangian. The validity of such a local construction is justified by the fact that the final Hamiltonian equations do really imply the initial Lagrangian equations.

Consider again a general Toda system. Assume that $t = z^0$ and $x = z^1$ are flat Minkowski coordinates on M . In these coordinates one has

$$\|\eta_{ij}\| = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The expression for the density of the Lagrangian (4.7) takes the form

$$\mathcal{L} = \frac{1}{2} \kappa h_{\mu\nu}(\xi) \partial_t \xi^\mu \partial_t \xi^\nu - \frac{1}{2} \kappa h_{\mu\nu}(\xi) \partial_x \xi^\mu \partial_x \xi^\nu - \kappa \lambda_{\mu\nu}(\xi) \partial_t \xi^\mu \partial_x \xi^\nu - \kappa V(\xi).$$

Here and below we denote $\partial_t = \partial/\partial t$ and $\partial_x = \partial/\partial x$. The density of the energy functional is

$$\mathcal{E} = \frac{\partial \mathcal{L}}{\partial(\partial_t \xi^\mu)} \partial_t \xi^\mu - \mathcal{L} = \frac{\kappa}{2} h_{\mu\nu}(\xi) \partial_t \xi^\mu \partial_t \xi^\nu + \frac{\kappa}{2} h_{\mu\nu}(\xi) \partial_x \xi^\mu \partial_x \xi^\nu + \kappa V(\xi). \tag{5.1}$$

For the generalized momenta one has the expression

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_t \xi^\mu)} = \kappa h_{\mu\nu}(\xi) \partial_t \xi^\nu - \kappa \lambda_{\mu\nu}(\xi) \partial_x \xi^\nu.$$

We can write the inverse relation which expresses the generalized velocities $\partial_t \xi^\mu$ via the generalized momenta:

$$\partial_t \xi^\mu = \frac{1}{\kappa} h^{\mu\nu}(\xi) [\pi_\nu + \kappa \lambda_{\nu\rho}(\xi) \partial_x \xi^\rho],$$

where

$$h^{\mu\rho}(y) h_{\rho\nu}(y) = \delta_\nu^\mu.$$

Substituting the above expression for $\partial_t \xi^\mu$ into the relation (5.1), we obtain for the density of the Hamiltonian the following expression:

$$\begin{aligned} \mathcal{H} = & \frac{1}{2\kappa} h^{\mu\nu}(\xi) [\pi_\mu + \kappa \lambda_{\mu\rho}(\xi) \partial_x \xi^\rho] [\pi_\nu + \kappa \lambda_{\nu\sigma}(\xi) \partial_x \xi^\sigma] \\ & + \frac{\kappa}{2} h_{\mu\nu}(\xi) \partial_x \xi^\mu \partial_x \xi^\nu + \kappa V(\xi). \end{aligned}$$

Recall that the non-vanishing Poisson brackets for the fields ξ^μ and the generalized momenta π_μ have the form

$$\{\xi^\mu(x), \pi_\nu(x')\} = \delta_\nu^\mu \delta(x - x').$$

Using this relation, one can write the Hamiltonian equations of motion and prove that they are equivalent to the Lagrangian equations of motion.

The phase space of the system is described by the fields ξ^μ and the generalized momenta π_μ . They depend on the choice of the coordinates y^μ in G_0 . To describe the phase space in terms independent of this choice, consider first the quantities

$$j_\alpha = -X_\alpha^\mu(\xi) [\pi_\mu + \kappa \lambda_{\mu\rho}(\xi) \partial_x \xi^\rho] + \kappa c_{\alpha\gamma} \theta_\rho^\gamma(\xi) \partial_x \xi^\rho,$$

where the functions $X_\alpha^\mu(y)$ are defined by (B.7). As is shown in Appendix C, the Poisson brackets for the quantities $j_\alpha(x)$ are

$$\{j_\alpha(x), j_\beta(x')\} = j_\gamma(x) f_{\alpha\beta}^\gamma \delta(x - x') - 2\kappa c_{\alpha\beta} \delta'(x - x'). \tag{5.2}$$

Thus, we have a realization of the so-called current algebra.

It is also convenient to consider the quantities

$$\bar{j}_\alpha = -\bar{X}_\alpha^\mu(\xi) [\pi_\mu + \kappa \lambda_{\mu\rho}(\xi) \partial_x \xi^\rho] - \kappa c_{\alpha\gamma} \bar{\theta}_\rho^\gamma(\xi) \partial_x \xi^\rho,$$

where the functions $\bar{\theta}_\mu^\alpha(y)$ are the components of the right-invariant Maurer–Cartan form of G_0 and the functions $\bar{X}_\alpha^\mu(y)$ are defined by the equality (B.11). One can show that

$$\{\bar{j}_\alpha(x), \bar{j}_\beta(x')\} = -\bar{j}_\gamma(x) f_{\alpha\beta}^\gamma \delta(x - x') + 2\kappa c_{\alpha\beta} \delta'(x - x') \tag{5.3}$$

and that

$$\{j_\alpha(x), \bar{j}_\beta(x')\} = 0. \tag{5.4}$$

The main relation used here is the equality (C.4).

Now, using the definition (4.6) of $h_{\mu\nu}(y)$, we obtain

$$h^{\mu\nu}(y) = X_\alpha^\mu(y) c^{\alpha\beta} X_\beta^\nu(y),$$

where

$$c^{\alpha\gamma} c_{\gamma\beta} = \delta_\beta^\alpha.$$

The above equality allows us to demonstrate that

$$\begin{aligned} c^{\alpha\beta} j_\alpha j_\beta &= h_{\mu\nu}(\xi) [\pi_\mu + \kappa \lambda_{\mu\rho}(\xi) \partial_x \xi^\rho] [\pi_\nu + \kappa \lambda_{\nu\sigma}(\xi) \partial_x \xi^\sigma] \\ &\quad - 2\kappa \partial_x \xi^\mu [\pi_\mu + \kappa \lambda_{\mu\rho}(\xi) \partial_x \xi^\rho] + \kappa^2 h_{\mu\nu}(\xi) \partial_x \xi^\mu \partial_x \xi^\nu. \end{aligned}$$

Further, the relation (B.20) leads to another representation of $h_{\mu\nu}(y)$ and $h^{\mu\nu}(y)$:

$$h_{\mu\nu}(y) = \bar{\theta}_\mu^\alpha(y) c_{\alpha\beta} \bar{\theta}_\nu^\beta(y), \quad h^{\mu\nu}(y) = \bar{X}_\alpha^\mu(y) c^{\alpha\beta} \bar{X}_\beta^\nu(y).$$

Using these relations we find

$$\begin{aligned} c^{\alpha\beta} \bar{j}_\alpha \bar{j}_\beta &= h_{\mu\nu}(\xi) [\pi_\mu + \kappa \lambda_{\mu\rho}(\xi) \partial_x \xi^\rho] [\pi_\nu + \kappa \lambda_{\nu\sigma}(\xi) \partial_x \xi^\sigma] \\ &\quad + 2\kappa \partial_x \xi^\mu [\pi_\mu + \kappa \lambda_{\mu\rho}(\xi) \partial_x \xi^\rho] + \kappa^2 h_{\mu\nu}(\xi) \partial_x \xi^\mu \partial_x \xi^\nu. \end{aligned}$$

It becomes clear that the density of the Hamiltonian \mathcal{H} can be written in the Sugawara form [34,35]

$$\mathcal{H} = \frac{1}{4\kappa} [c^{\alpha\beta} j_\alpha j_\beta + c^{\alpha\beta} \bar{j}_\alpha \bar{j}_\beta] + \kappa V(\xi). \tag{5.5}$$

The quantities j_α and \bar{j}_α do not depend on the choice of coordinates y^μ in the Lie group G_0 but they depend on the choice of the basis $\{e_\alpha\}$. To get rid of this dependence introduce the matrix-valued quantities

$$j = e_\alpha c^{\alpha\beta} j_\beta, \quad \bar{j} = e_\alpha c^{\alpha\beta} \bar{j}_\beta.$$

Note that j and \bar{j} are the Hamiltonian counterparts of the quantities $-\kappa\gamma^{-1}\partial_-\gamma$ and $-\kappa\partial_+\gamma\gamma^{-1}$, respectively.

Our next task is to rewrite the relations (5.2)–(5.4) in terms of Poisson brackets of the matrix-valued quantities j and \bar{j} . Actually we will consider j and \bar{j} as functionals on the phase space of the system taking values in the associative algebra $\text{Mat}_N(\mathbb{R})$. The corresponding definition of the Poisson bracket for algebra-valued functionals on a phase space and its main properties are discussed in Appendix D.

Consider the element $C \in \mathfrak{g}_0 \otimes \mathfrak{g}_0$ defined as

$$C = e_\alpha \otimes e_\beta c^{\alpha\beta}. \tag{5.6}$$

Introducing the notation

$$e^\alpha = e_\beta c^{\beta\alpha},$$

we can write

$$C = e^\alpha \otimes e^\beta c_{\alpha\beta} = e^\alpha \otimes e_\alpha = e_\alpha \otimes e^\alpha.$$

Using the relation

$$[C, e^\gamma \otimes I_N] = [I_N \otimes e^\gamma, C] = f_{\alpha\beta}^\gamma e^\alpha \otimes e^\beta,$$

we obtain

$$\{j(x) \otimes j(x')\} = -[C, I_N \otimes j(x)] \delta(x - x') - 2\kappa C \delta'(x - x'), \tag{5.7}$$

$$\{\bar{j}(x) \otimes \bar{j}(x')\} = [C, I_N \otimes \bar{j}(x)] \delta(x - x') + 2\kappa C \delta'(x - x'), \tag{5.8}$$

$$\{j(x) \otimes \bar{j}(x')\} = 0. \tag{5.9}$$

Using the relation (B.13), we come to the equality

$$\{\gamma(x), j_\alpha(x')\} = -\gamma(x) e_\alpha \delta(x - x')$$

that gives

$$\{\gamma(x) \otimes j(x')\} = -(\gamma(x) \otimes I_N) C \delta(x - x'). \tag{5.10}$$

Similarly, the relation (B.14) implies

$$\{\gamma(x) \otimes \bar{j}(x')\} = -C(\gamma(x) \otimes I_N) \delta(x - x'). \tag{5.11}$$

It is also clear that

$$\{\gamma(x) \otimes \gamma(x')\} = 0.$$

Taking into account (5.5), we obtain

$$\mathcal{H} = \frac{1}{4\kappa} [B(j, j) + B(\bar{j}, \bar{j})] - \kappa B(a_-, \gamma^{-1} a_+ \gamma).$$

It is not difficult to write down the corresponding Hamiltonian equations. If we choose as the basis quantities describing the phase space of the system the quantities γ and j , we come to the equations

$$\partial_t \gamma = \partial_x \gamma - \frac{1}{\kappa} \gamma j, \quad \partial_t j = -\partial_x j - \kappa [a_-, \gamma^{-1} a_+ \gamma]. \tag{5.12}$$

In the case when the quantities γ and \bar{j} are chosen as the basis quantities, one obtains

$$\partial_t \gamma = -\partial_x \gamma - \frac{1}{\kappa} \bar{j} \gamma, \quad \partial_t \bar{j} = \partial_x \bar{j} - \kappa [\gamma a_- \gamma^{-1}, a_+]. \tag{5.13}$$

It is clear that the obtained Hamiltonian equations are equivalent to the Toda equations (2.2) and (2.3), respectively.

6. W-algebra

In this section we return again to the Toda system defined in Section 2 and find the Poisson brackets for the characteristic integrals given in Section 3. Recall that the Lie group G_0 in the case under consideration is isomorphic to the direct product of two copies of the Lie group $GL_n(\mathbb{R})$, and the Lie algebra \mathfrak{g}_0 is isomorphic to the direct product of two copies of the Lie algebra $\mathfrak{gl}_n(\mathbb{R})$.

The standard basis of the Lie algebra $\mathfrak{gl}_n(\mathbb{R})$ consists of the matrices $e_i^j, i, j = 1, \dots, n$, defined as

$$(e_i^j)_l^k = \delta_i^k \delta_l^j.$$

Certainly, these matrices form a basis of the algebra $Mat_n(\mathbb{R})$, too. The main property of these matrices is provided by the relation

$$e_i^j e_k^l = e_i^l \delta_k^j.$$

Using this relation and the equality

$$\text{tr}(e_i^j) = \delta_i^j,$$

one obtains

$$\text{tr}(e_i^j e_k^l) = \delta_i^l \delta_k^j.$$

A natural basis of the Lie algebra \mathfrak{g}_0 is formed by the matrices

$$E_i^{(1)j} = \begin{pmatrix} e_i^j & 0 \\ 0 & 0 \end{pmatrix}, \quad E_i^{(2)j} = \begin{pmatrix} 0 & 0 \\ 0 & e_i^j \end{pmatrix}, \quad i, j = 1, \dots, n.$$

Recall that we assumed the Lie algebra \mathfrak{g}_0 in the case under consideration to be equipped with the scalar product B defined by the relation (4.12). Therefore, we have

$$B(E_i^{(r)j}, E_k^{(s)l}) = \text{tr}(E_i^{(r)j} E_k^{(s)l}) = \delta_i^l \delta_k^j \delta^{rs}.$$

From the natural block matrix structure of the space $\mathfrak{g}_0 \otimes \mathfrak{g}_0$, we see that the element C introduced by (5.6) has in our case the form

$$C = \begin{pmatrix} C_n & 0 \\ 0 & C_n \end{pmatrix}, \tag{6.1}$$

where the element $C_n \in \mathfrak{gl}_n(\mathbb{R}) \otimes \mathfrak{gl}_n(\mathbb{R})$ is defined by the relation

$$C_n = e_i^j \otimes e_j^i.$$

One can verify the validity of the equalities

$$C_n(e_i^j \otimes e_k^l) = e_k^j \otimes e_i^l, \quad (e_i^j \otimes e_k^l)C_n = e_i^l \otimes e_k^j.$$

These imply that the action of the permutation operator P on $\mathfrak{gl}_n(\mathbb{R}) \otimes \mathfrak{gl}_n(\mathbb{R})$, or on $\text{Mat}_n(\mathbb{R}) \otimes \text{Mat}_n(\mathbb{R})$, can be realized with the help of the element C as

$$P(a \otimes b) = C_n(a \otimes b)C_n.$$

Note also that

$$C_n^2 = I_n \otimes I_n.$$

It is quite natural to use for the mapping γ the parameterization (2.8) and for the quantities j and \bar{j} the parameterizations

$$j = \begin{pmatrix} \mathcal{J}^{(1)} & 0 \\ 0 & \mathcal{J}^{(2)} \end{pmatrix}, \quad \bar{j} = \begin{pmatrix} \bar{\mathcal{J}}^{(1)} & 0 \\ 0 & \bar{\mathcal{J}}^{(2)} \end{pmatrix},$$

where the functions $\mathcal{J}^{(r)}$ and $\bar{\mathcal{J}}^{(r)}$, $r = 1, 2$, take values in $\mathfrak{gl}_n(\mathbb{R})$. It is clear that the relations (5.7)–(5.9) can be written now as

$$\{\mathcal{J}^{(r)}(x) \otimes \mathcal{J}^{(s)}(x')\} = -([C_n, I_n \otimes \mathcal{J}^{(r)}(x)] \delta(x - x') + 2\kappa C_n \delta'(x - x')) \delta^{rs}, \tag{6.2}$$

$$\{\bar{\mathcal{J}}^{(r)}(x) \otimes \bar{\mathcal{J}}^{(s)}(x')\} = ([C_n, I_n \otimes \bar{\mathcal{J}}^{(r)}(x)] \delta(x - x') + 2\kappa C_n \delta'(x - x')) \delta^{rs}, \tag{6.3}$$

$$\{\mathcal{J}^{(r)}(x) \otimes \bar{\mathcal{J}}^{(s)}(x')\} = 0, \tag{6.4}$$

and the relations (5.10) and (5.11) take the forms

$$\{\Gamma^{(r)}(x) \otimes \mathcal{J}^{(s)}(x')\} = -(\Gamma^{(r)}(x) \otimes I_n) C_n \delta(x - x') \delta^{rs}, \tag{6.5}$$

$$\{\Gamma^{(r)}(x) \otimes \bar{\mathcal{J}}^{(s)}(x')\} = -C_n (\Gamma^{(r)}(x) \otimes I_n) \delta(x - x') \delta^{rs}. \tag{6.6}$$

The Hamiltonian equations of motion (5.12) are now of the forms

$$\begin{aligned} \partial_t \Gamma^{(1)} &= \partial_x \Gamma^{(1)} - \frac{1}{\kappa} \Gamma^{(1)} \mathcal{J}^{(1)}, & \partial_t \mathcal{J}^{(1)} &= -\partial_x \mathcal{J}^{(1)} + \kappa \Gamma^{(1)-1} \Gamma^{(2)}, \\ \partial_t \Gamma^{(2)} &= \partial_x \Gamma^{(2)} - \frac{1}{\kappa} \Gamma^{(2)} \mathcal{J}^{(2)}, & \partial_t \mathcal{J}^{(2)} &= -\partial_x \mathcal{J}^{(2)} - \kappa \Gamma^{(1)-1} \Gamma^{(2)}, \end{aligned}$$

while for Eqs. (5.13) we have

$$\begin{aligned} \partial_t \Gamma^{(1)} &= -\partial_x \Gamma^{(1)} - \frac{1}{\kappa} \bar{\mathcal{J}}^{(1)} \Gamma^{(1)}, & \partial_t \bar{\mathcal{J}}^{(1)} &= \partial_x \bar{\mathcal{J}}^{(1)} + \kappa \Gamma^{(2)} \Gamma^{(1)-1}, \\ \partial_t \Gamma^{(2)} &= -\partial_x \Gamma^{(2)} - \frac{1}{\kappa} \bar{\mathcal{J}}^{(2)} \Gamma^{(2)}, & \partial_t \bar{\mathcal{J}}^{(2)} &= \partial_x \bar{\mathcal{J}}^{(2)} - \kappa \Gamma^{(2)} \Gamma^{(1)-1}. \end{aligned}$$

Let us find Hamiltonian counterparts of the characteristic integrals W_1 and W_2 defined by the relation (3.8). There is no problem with the characteristic integral W_1 . Its Hamiltonian counterpart obviously is

$$\mathcal{W}_1 = \mathcal{J}^{(1)} + \mathcal{J}^{(2)}. \tag{6.7}$$

The characteristic integral W_2 contains higher time derivatives and has no direct Hamiltonian counterpart. However, here one can use the fact that characteristic integrals are defined up to terms vanishing at the equations of motion. Therefore, one can use the equations of motion to get equivalent characteristic integrals which do not contain higher time derivatives.

For the case under consideration, using the definition (3.7), the equations of motion (2.9) and the equality

$$\partial_- = \partial_+ - 2\partial_x,$$

we obtain

$$\frac{1}{2}(\partial_- \Sigma_-^{(1)} - \partial_- \Sigma_-^{(2)}) = -\Gamma^{(1)-1} \Gamma^{(2)} - (\partial_x \Sigma_-^{(1)} - \partial_x \Sigma_-^{(2)}).$$

Hence, the Hamiltonian counterpart of the characteristic integral W_2 is

$$\mathcal{W}_2 = \mathcal{J}^{(1)} \mathcal{J}^{(2)} - \kappa(\partial_x \mathcal{J}^{(1)} - \partial_x \mathcal{J}^{(2)}) + \kappa^2 \Gamma^{(1)-1} \Gamma^{(2)}. \tag{6.8}$$

The Poisson bracket for the characteristic integral \mathcal{W}_1 follows directly from (6.2)

$$\{\mathcal{W}_1(x) \otimes \mathcal{W}_1(x')\} = -[C_n, I_n \otimes \mathcal{W}_1(x)] \delta(x - x') - 4\kappa C_n \delta'(x - x'). \tag{6.9}$$

The calculations needed to obtain expressions for other Poisson brackets are more complicated. The main formulas are presented in Appendix E. The final result is

$$\begin{aligned} \{\mathcal{W}_1(x) \otimes \mathcal{W}_2(x')\} &= -[C_n, I_n \otimes \mathcal{W}_2(x)] \delta(x - x') \\ &\quad - \kappa [C_n, I_n \otimes \mathcal{W}_1(x')]_+ \delta'(x - x'), \end{aligned} \tag{6.10}$$

$$\begin{aligned} &\{\mathcal{W}_2(x) \otimes \mathcal{W}_2(x')\} \\ &= (I_n \otimes \mathcal{W}_2(x)) C_n (I_n \otimes \mathcal{W}_1(x)) \delta(x - x') \\ &\quad - (I_n \otimes \mathcal{W}_1(x)) C_n (I_n \otimes \mathcal{W}_2(x)) \delta(x - x') \\ &\quad - \frac{1}{2} \kappa^2 [C_n, I_n \otimes \partial_x^2 \mathcal{W}_1(x)] \delta(x - x') \\ &\quad + \kappa [C_n, I_n \otimes (\mathcal{W}_2(x) + \mathcal{W}_2(x'))]_+ \delta'(x - x') \\ &\quad - \kappa (I_n \otimes \mathcal{W}_1(x)) C_n (I_n \otimes \mathcal{W}_1(x)) \delta'(x - x') \\ &\quad - \kappa (I_n \otimes \mathcal{W}_1(x')) C_n (I_n \otimes \mathcal{W}_1(x')) \delta'(x - x') \\ &\quad + \frac{3}{2} \kappa^2 [C_n, I_n \otimes (\mathcal{W}_1(x) + \mathcal{W}_1(x'))] \delta''(x - x') + 4\kappa^3 C_n \delta'''(x - x'). \end{aligned} \tag{6.11}$$

Some terms of the last formula can be combined into a commutator. Actually it would give a more compact expression. Nevertheless, we prefer to use the above form of the expression which is more convenient for the comparison with the case considered in the next section. After some redefinitions one can get convinced that the obtained expressions for the Poisson brackets of the characteristic integrals coincide with the expressions obtained via the method of Hamiltonian reduction [36]. Our direct rederivation of these expressions can be considered, in particular, as the verification needed by the reasons given in Section 1.

The Hamiltonian counterparts of the characteristic integrals \bar{W}_1 and \bar{W}_2 are

$$\bar{W}_1 = \bar{\mathcal{J}}^{(1)} + \bar{\mathcal{J}}^{(2)}, \tag{6.12}$$

$$\bar{W}_2 = \bar{\mathcal{J}}^{(2)} \bar{\mathcal{J}}^{(1)} + \kappa(\partial_x \bar{\mathcal{J}}^{(1)} - \partial_x \bar{\mathcal{J}}^{(2)}) + \kappa^2 \Gamma^{(2)} \Gamma^{(1)-1}, \tag{6.13}$$

and the Poisson brackets for them look as

$$\{\bar{W}_1(x) \otimes \bar{W}_1(x')\} = [C_n, I_n \otimes \bar{W}_1(x)] \delta(x - x') + 4\kappa C_n \delta'(x - x'), \tag{6.14}$$

$$\{\bar{W}_1(x) \otimes \bar{W}_2(x')\} = [C_n, I_n \otimes \bar{W}_2(x)] \delta(x - x') + \kappa [C_n, I_n \otimes \bar{W}_1(x')]_+ \delta'(x - x'), \tag{6.15}$$

$$\begin{aligned} & \{\bar{W}_2(x) \otimes \bar{W}_2(x')\} \\ &= - (I_n \otimes \bar{W}_2(x)) C_n (I_n \otimes \bar{W}_1(x)) \delta(x - x') \\ &+ (I_n \otimes \bar{W}_1(x)) C_n (I_n \otimes \bar{W}_2(x)) \delta(x - x') \\ &+ \frac{1}{2} \kappa^2 [C_n, I_n \otimes \partial_x^2 \bar{W}_1(x)] \delta(x - x') \\ &- \kappa [C_n, I_n \otimes (\bar{W}_2(x) + \bar{W}_2(x'))]_+ \delta'(x - x') \\ &+ \kappa (I_n \otimes \bar{W}_1(x)) C_n (I_n \otimes \bar{W}_1(x)) \delta'(x - x') \\ &+ \kappa (I_n \otimes \bar{W}_1(x')) C_n (I_n \otimes \bar{W}_1(x')) \delta'(x - x') \\ &- \frac{3}{2} \kappa^2 [C_n, I_n \otimes (\bar{W}_1(x) + \bar{W}_1(x'))] \delta''(x - x') - 4\kappa^3 C_n \delta'''(x - x'). \end{aligned} \tag{6.16}$$

Here we again write the result of our calculations in the form which is convenient from the point of view of the example considered in the next section.

Let us find the Poisson bracket for the Hamiltonian counterparts \mathcal{T}'_{--} and \mathcal{T}'_{++} of the components T'_{--} and T'_{++} of the energy–momentum tensor. It is known that they are the generators of the conformal transformations.

As follows from (4.13), one has

$$\mathcal{T}'_{--} = \frac{1}{\kappa} \text{tr}[\mathcal{W}_1^2 - 2\mathcal{W}_2], \quad \mathcal{T}'_{++} = \frac{1}{\kappa} \text{tr}[\bar{\mathcal{W}}_1^2 - 2\bar{\mathcal{W}}_2].$$

To find the Poisson brackets in question, we start with the relation

$$\begin{aligned} \{\mathcal{W}_1^2(x) \otimes \mathcal{W}_1(x')\} &= -[C_n, I_n \otimes \mathcal{W}_1^2(x)] \delta(x - x') \\ &- 4\kappa [C_n, I_n \otimes \mathcal{W}_1(x)]_+ \delta'(x - x'). \end{aligned} \tag{6.17}$$

This relation gives

$$\begin{aligned} & \{\mathcal{W}_1^2(x) \otimes \mathcal{W}_1^2(x')\} \\ &= -[C_n, I_n \otimes \mathcal{W}_1^3(x)] \delta(x - x') - [C_n, \mathcal{W}_1(x) \otimes \mathcal{W}_1^2(x)] \delta(x - x') \\ &\quad - 2\kappa[C_n, I_n \otimes (\mathcal{W}_1(x) \partial_x \mathcal{W}_1(x) - \partial_x \mathcal{W}_1(x) \mathcal{W}_1(x))] \delta(x - x') \\ &\quad - 2\kappa[C_n, I_n \otimes (\mathcal{W}_1^2(x) + \mathcal{W}_1^2(x'))]_+ \delta'(x - x') \\ &\quad - 4\kappa C_n (\mathcal{W}_1(x) \otimes \mathcal{W}_1(x) + \mathcal{W}_1(x') \otimes \mathcal{W}_1(x')) \delta'(x - x'). \end{aligned} \tag{6.18}$$

For any $\text{Mat}_n(\mathbb{R})$ -valued functionals F and G one obtains

$$\{\text{tr } F, \text{tr } G\} = \text{tr}\{F \otimes G\}.$$

Besides, for any $a, b \in \text{Mat}_n(\mathbb{R})$ one has

$$\text{tr}(a \otimes b) = \text{tr } a \text{tr } b,$$

and

$$\text{tr}(C_n(a \otimes b)) = \text{tr}((a \otimes b)C_n) = \text{tr}(ab).$$

Using these relations, we obtain from (6.18) the equality

$$\{\text{tr } \mathcal{W}_1^2(x), \text{tr } \mathcal{W}_1^2(x')\} = -8\kappa(\text{tr } \mathcal{W}_1^2(x) + \text{tr } \mathcal{W}_1^2(x')) \delta'(x - x'). \tag{6.19}$$

The relation

$$\begin{aligned} \{\mathcal{W}_1^2(x) \otimes \mathcal{W}_2(x')\} &= -[C_n, \mathcal{W}_1(x) \otimes \mathcal{W}_2(x)] \delta(x - x') \\ &\quad - C_n(I_n \otimes \mathcal{W}_1(x) \mathcal{W}_2(x) - \mathcal{W}_2(x) \mathcal{W}_1(x) \otimes I_n) \delta(x - x') \\ &\quad - \kappa[C_n, \mathcal{W}_1(x) \otimes \mathcal{W}_1(x')]_+ \delta'(x - x') \\ &\quad - \kappa C_n(I_n \otimes \mathcal{W}_1(x) \mathcal{W}_1(x') + \mathcal{W}_1(x') \mathcal{W}_1(x) \otimes I_n) \delta'(x - x') \end{aligned} \tag{6.20}$$

helps us to obtain that

$$\{\text{tr } \mathcal{W}_1^2(x), \text{tr } \mathcal{W}_2(x')\} = -2\kappa(\text{tr } \mathcal{W}_1^2(x) + \text{tr } \mathcal{W}_1^2(x')) \delta'(x - x'). \tag{6.21}$$

Further, the relation (6.11) gives

$$\begin{aligned} \{\text{tr } \mathcal{W}_2(x), \text{tr } \mathcal{W}_2(x')\} &= 2\kappa(\text{tr } \mathcal{W}_2(x) + \text{tr } \mathcal{W}_2(x')) \delta'(x - x') \\ &\quad - \kappa(\text{tr } \mathcal{W}_1^2(x) + \text{tr } \mathcal{W}_1^2(x')) \delta'(x - x') + 4\kappa^3 n \delta'''(x - x'). \end{aligned} \tag{6.22}$$

Taking into account the relations (6.19), (6.21) and (6.22) we get

$$\{\mathcal{T}'_{--}(x), \mathcal{T}'_{--}(x')\} = -4(\mathcal{T}'_{--}(x) + \mathcal{T}'_{--}(x')) \delta'(x - x') + 16\kappa n \delta'''(x - x').$$

In a similar way we come to the equality

$$\{\mathcal{T}'_{++}(x), \mathcal{T}'_{++}(x')\} = 4(\mathcal{T}'_{++}(x) + \mathcal{T}'_{++}(x')) \delta'(x - x') - 16\kappa n \delta'''(x - x'),$$

and it is evident that

$$\{\mathcal{T}'_{--}(x), \mathcal{T}'_{++}(x')\} = 0.$$

It is clear from these relations that the quantities

$$\mathcal{V}(x) = \frac{1}{4}\mathcal{T}'_{--}(x), \quad \bar{\mathcal{V}}(x) = \frac{1}{4}\mathcal{T}'_{++}(x) \tag{6.23}$$

are generators of two copies of the Virasoro algebra:

$$\{\mathcal{V}(x), \mathcal{V}(x')\} = -(\mathcal{V}(x) + \mathcal{V}(x')) \delta'(x - x') + \kappa n \delta'''(x - x'), \tag{6.24}$$

$$\{\bar{\mathcal{V}}(x), \bar{\mathcal{V}}(x')\} = (\bar{\mathcal{V}}(x) + \bar{\mathcal{V}}(x')) \delta'(x - x') - \kappa n \delta'''(x - x'), \tag{6.25}$$

$$\{\mathcal{V}(x), \bar{\mathcal{V}}(x')\} = 0. \tag{6.26}$$

The generators $\mathcal{V}(x)$ and $\bar{\mathcal{V}}(x)$ produce infinitesimal conformal transformations via the following standard procedure. Let us define

$$\mathcal{V}_\varepsilon(t) = \int dx \varepsilon(t, x) \mathcal{V}(t, x),$$

where ε is an arbitrary infinitesimal function on M which satisfies the relation

$$\partial_+ \varepsilon = \partial_t \varepsilon + \partial_x \varepsilon = 0.$$

Actually, \mathcal{V}_ε is an integrated characteristic integral, therefore, it does not depend on t . Consider the infinitesimal transformations defined for an arbitrary observable $F(t)$ as

$$\delta F(t) = \{\mathcal{V}_\varepsilon(t), F(t)\}.$$

It can be shown that these transformations are the infinitesimal version of the conformal transformations described by the relation (2.13) and (2.14) with $\zeta^+(z^+) = z^+$. Similarly, the quantities

$$\bar{\mathcal{V}}_{\bar{\varepsilon}}(t) = \int dx \bar{\varepsilon}(t, x) \bar{\mathcal{V}}(t, x),$$

where

$$\partial_- \bar{\varepsilon} = \partial_t \bar{\varepsilon} - \partial_x \bar{\varepsilon} = 0$$

generate the infinitesimal conformal transformations described by the relation (2.13) and (2.14) with $\zeta^-(z^-) = z^-$.

Now we will find the conformal weights of \mathcal{W}_1 and \mathcal{W}_2 . Recall that a field $\Phi(x)$ has the conformal weight h with respect to $\mathcal{V}(x)$ if

$$\{\mathcal{V}(x), \Phi(x')\} = -(\Phi(x) + (h - 1)\Phi(x')) \delta'(x - x') + \dots ,$$

where the dots stand for possible central terms, and it has the conformal weight \bar{h} with respect to $\bar{\mathcal{V}}(x)$ if

$$\{\bar{\mathcal{V}}(x), \Phi(x')\} = (\Phi(x) + (\bar{h} - 1)\Phi(x')) \delta'(x - x') + \dots .$$

Introduce two mappings tr_1 and tr_2 from $\text{Mat}_n(\mathbb{R}) \otimes \text{Mat}_n(\mathbb{R})$ to $\text{Mat}_n(\mathbb{R})$ given by

$$\text{tr}_1(a \otimes b) = (\text{tr } a)b, \quad \text{tr}_2(a \otimes b) = a(\text{tr } b).$$

It can be verified that these mappings satisfy the relations

$$\begin{aligned} \text{tr}_1(C_n(a \otimes b)) &= ab, & \text{tr}_1((a \otimes b)C_n) &= ba, \\ \text{tr}_2(C_n(a \otimes b)) &= ba, & \text{tr}_2((a \otimes b)C_n) &= ab. \end{aligned}$$

For any $\text{Mat}_n(\mathbb{R})$ -valued functionals F and G one has

$$\text{tr}_1\{F \otimes G\} = \{\text{tr } F \otimes G\}, \quad \text{tr}_2\{F \otimes G\} = \{F \otimes \text{tr } G\}.$$

Using the above relations, one obtains from (6.17) and (6.10) the equalities

$$\begin{aligned} \{\text{tr } \mathcal{W}_1^2(x) \otimes \mathcal{W}_1(x')\} &= -8\kappa \mathcal{W}_1(x) \delta(x - x'), \\ \{\text{tr } \mathcal{W}_2(x) \otimes \mathcal{W}_1(x')\} &= -2\kappa \mathcal{W}_1(x) \delta(x - x'). \end{aligned}$$

Hence, we come to the relation

$$\{\mathcal{V}(x) \otimes \mathcal{W}_1(x')\} = -\mathcal{W}_1(x) \delta'(x - x').$$

From the relation (6.20) we obtain

$$\begin{aligned} \{\text{tr } \mathcal{W}_1^2(x) \otimes \mathcal{W}_2(x')\} &= -2(\mathcal{W}_1(x)\mathcal{W}_2(x) - \mathcal{W}_2(x)\mathcal{W}_1(x)) \delta(x - x') \\ &\quad - 2\kappa(\mathcal{W}_1^2(x) + \mathcal{W}_1^2(x')) \delta'(x - x'), \end{aligned}$$

and the equality (6.11) gives

$$\begin{aligned} \{\text{tr } \mathcal{W}_2(x) \otimes \mathcal{W}_2(x')\} &= -(\mathcal{W}_1(x)\mathcal{W}_2(x) - \mathcal{W}_2(x)\mathcal{W}_1(x)) \delta(x - x') - \kappa(\mathcal{W}_1^2(x) + \mathcal{W}_1^2(x')) \delta'(x - x') \\ &\quad + 2\kappa(\mathcal{W}_2(x) + \mathcal{W}_2(x')) \delta'(x - x') + 4\kappa^3 I_n \delta'''(x - x'). \end{aligned}$$

Consequently, one has

$$\{\mathcal{V}(x) \otimes \mathcal{W}_2(x')\} = -(\mathcal{W}_2(x) + \mathcal{W}_2(x')) \delta'(x - x') - 2\kappa^2 \delta'''(x - x').$$

Thus, the characteristic integral \mathcal{W}_1 has the conformal weight 1 and the characteristic integral \mathcal{W}_2 has the conformal weight 2 with respect to $\mathcal{V}(x)$. Similarly, we obtain that the characteristic integral $\bar{\mathcal{W}}_1$ has the conformal weight 1 and the characteristic integral $\bar{\mathcal{W}}_2$ has the conformal weight 2 with respect to $\bar{\mathcal{V}}(x)$.

In the end of this section we find the form of the infinitesimal symmetry transformations generated by the characteristic integrals. First consider the quantity

$$\mathcal{W}_\varepsilon(t) = \int dx \operatorname{tr}[\varepsilon_1(t, x)\mathcal{W}_1(t, x) + \varepsilon_2(t, x)\mathcal{W}_2(t, x)], \tag{6.27}$$

where ε_1 and ε_2 are arbitrary infinitesimal matrix-valued functions on M satisfying the relations

$$\partial_+ \varepsilon_1 = \partial_t \varepsilon_1 + \partial_x \varepsilon_1 = 0, \quad \partial_+ \varepsilon_2 = \partial_t \varepsilon_2 + \partial_x \varepsilon_2 = 0.$$

The infinitesimal transformations generated by $\mathcal{W}_\varepsilon(t)$ written in the Lagrangian form are

$$\delta_\varepsilon \Gamma^{(1)} = \Gamma^{(1)} \varepsilon_1 - \kappa \Gamma^{(1)} \Gamma^{(2)-1} \partial_- \Gamma^{(2)} \varepsilon_2 - \frac{1}{2} \kappa \Gamma^{(1)} \partial_- \varepsilon_2, \tag{6.28}$$

$$\delta_\varepsilon \Gamma^{(2)} = \Gamma^{(2)} \varepsilon_1 - \kappa \Gamma^{(2)} \varepsilon_2 \Gamma^{(1)-1} \partial_- \Gamma^{(1)} + \frac{1}{2} \kappa \Gamma^{(2)} \partial_- \varepsilon_2. \tag{6.29}$$

According to (6.9)–(6.11), the generators \mathcal{W}_ε satisfy the relation

$$\{\mathcal{W}_\mu, \mathcal{W}_\nu\} = \mathcal{W}_{\varepsilon(\mu, \nu)} + \mathcal{C}(\mu, \nu), \tag{6.30}$$

with the infinitesimal matrix-valued functions and the central extension term being

$$\begin{aligned} \varepsilon_1(\mu, \nu) &= [\mu_1, \nu_1] + \kappa([\partial_x \mu_1, \nu_2]_+ + [\mu_2, \partial_x \nu_1]_+) - \kappa(\partial_x \nu_2 \mathcal{W}_1 \mu_2 - \nu_2 \mathcal{W}_1 \partial_x \mu_2) \\ &\quad - \kappa^2([\partial_x^2 \mu_2, \nu_2] - [\partial_x \mu_2, \partial_x \nu_2] + [\mu_2, \partial_x^2 \nu_2]), \\ \varepsilon_2(\mu, \nu) &= [\mu_1, \nu_2] + [\mu_2, \nu_1] + (\mu_2 \mathcal{W}_1 \nu_2 - \nu_2 \mathcal{W}_1 \mu_2) \\ &\quad + \kappa([\mu_2, \partial_x \nu_2]_+ - [\partial_x \mu_2, \nu_2]_+), \\ \mathcal{C}(\mu, \nu) &= 4\kappa \int dx \operatorname{tr}(\partial_x \mu_1 \nu_1) - 4\kappa^3 \int dx \operatorname{tr}(\partial_x^3 \mu_2 \nu_2). \end{aligned}$$

We see that the non-linear terms of the W -algebra made the transformation parameters ε_1 and ε_2 depending on the Toda fields and their derivatives, although only through the characteristic integral \mathcal{W}_1 .

Similarly, introducing the quantity

$$\bar{\mathcal{W}}_{\bar{\varepsilon}}(t) = \int dx \operatorname{tr}[\bar{\varepsilon}_1(t, x)\bar{\mathcal{W}}_1(t, x) + \bar{\varepsilon}_2(t, x)\bar{\mathcal{W}}_2(t, x)], \tag{6.31}$$

where the infinitesimal matrix-valued functions $\bar{\varepsilon}_1$ and $\bar{\varepsilon}_2$ satisfy the relations

$$\partial_- \bar{\varepsilon}_1 = \partial_t \bar{\varepsilon}_1 - \partial_x \bar{\varepsilon}_1 = 0, \quad \partial_- \bar{\varepsilon}_2 = \partial_t \bar{\varepsilon}_2 - \partial_x \bar{\varepsilon}_2 = 0,$$

we come to the following expressions for the infinitesimal transformations

$$\delta_{\bar{\varepsilon}} \Gamma^{(1)} = \bar{\varepsilon}_1 \Gamma^{(1)} - \kappa \bar{\varepsilon}_2 \partial_+ \Gamma^{(2)} \Gamma^{(2)-1} \Gamma^{(1)} - \frac{1}{2} \kappa \partial_+ \bar{\varepsilon}_2 \Gamma^{(1)}, \tag{6.32}$$

$$\delta_{\bar{\varepsilon}} \Gamma^{(2)} = \bar{\varepsilon}_1 \Gamma^{(2)} - \kappa \partial_+ \Gamma^{(1)} \Gamma^{(1)-1} \bar{\varepsilon}_2 \Gamma^{(2)} + \frac{1}{2} \kappa \partial_+ \bar{\varepsilon}_2 \Gamma^{(2)}. \tag{6.33}$$

The generators $\bar{\mathcal{W}}_{\bar{\varepsilon}}$ give rise to a closed algebra of the form (6.30), that can be found from the relations (6.14)–(6.16). Actually, we have

$$\{\bar{\mathcal{W}}_{\bar{\mu}}, \bar{\mathcal{W}}_{\bar{\nu}}\} = \bar{\mathcal{W}}_{\bar{\varepsilon}(\bar{\mu}, \bar{\nu})} + \bar{\mathcal{C}}(\bar{\mu}, \bar{\nu}), \tag{6.34}$$

where

$$\begin{aligned} \bar{\varepsilon}_1(\bar{\mu}, \bar{v}) &= -[\bar{\mu}_1, \bar{v}_1] - \kappa([\partial_x \bar{\mu}_1, \bar{v}_2]_+ + [\bar{\mu}_2, \partial_x \bar{v}_1]_+) + \kappa(\partial_x \bar{v}_2 \bar{\mathcal{W}}_1 \bar{\mu}_2 - \bar{v}_2 \bar{\mathcal{W}}_1 \partial_x \bar{\mu}_2) \\ &\quad + \kappa^2([\partial_x^2 \bar{\mu}_2, \bar{v}_2] - [\partial_x \bar{\mu}_2, \partial_x \bar{v}_2] + [\bar{\mu}_2, \partial_x^2 \bar{v}_2]), \\ \bar{\varepsilon}_2(\bar{\mu}, \bar{v}) &= -[\bar{\mu}_1, \bar{v}_2] - [\bar{\mu}_2, \bar{v}_1] - (\bar{\mu}_2 \bar{\mathcal{W}}_1 \bar{v}_2 - \bar{v}_2 \bar{\mathcal{W}}_1 \bar{\mu}_2) \\ &\quad - \kappa([\bar{\mu}_2, \partial_x \bar{v}_2]_+ - [\partial_x \bar{\mu}_2, \bar{v}_2]_+), \\ \bar{\mathcal{C}}(\bar{\mu}, \bar{v}) &= -4\kappa \int dx \operatorname{tr}(\partial_x \bar{\mu}_1 \bar{v}_1) + 4\kappa^3 \int dx \operatorname{tr}(\partial_x^3 \bar{\mu}_2 \bar{v}_2). \end{aligned}$$

One can verify that the transformations (6.28), (6.29), (6.32) and (6.33) are symmetry transformations for the Toda system under consideration. Putting $\varepsilon_2 = 0$ and $\bar{\varepsilon}_2 = 0$ we obtain the infinitesimal version of the transformations described by the relation (2.11).

7. Non-abelian Liouville equation

In this section we consider an example of a non-abelian Toda system associated with the Lie group $\operatorname{Sp}_n(\mathbb{R})$. It is convenient for our purposes to define this Lie group as a subgroup of the Lie group $\operatorname{GL}_{2n}(\mathbb{R})$ formed by all matrices $a \in \operatorname{GL}_{2n}(\mathbb{R})$ which satisfy the relation

$$a^t K_n a = K_n,$$

where the $2n \times 2n$ matrix K_n has the form

$$K_n = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}$$

with J_n being the skew-diagonal unit $n \times n$ matrix. The superscript t as usually means the transposition.

The Lie algebra $\mathfrak{sp}_n(\mathbb{R})$ of the Lie group $\operatorname{Sp}_n(\mathbb{R})$ is formed by all real $2n \times 2n$ matrices x which satisfy the relation

$$x^t K_n + K_n x = 0.$$

Using for a general $2n \times 2n$ matrix x block representation (2.4), we see that the above relation is equivalent to the equalities

$$x_{11}^T = -x_{22}, \quad x_{12}^T = x_{12}, \quad x_{21}^T = x_{21},$$

where for an $n \times n$ matrix x we have denoted

$$x^T = (J_n)^{-1} x^t J_n = J_n x^t J_n.$$

Actually the matrix x^T is the transpose of the matrix x with respect to the main skew diagonal.

Note that the matrix q defined by (2.5) belongs to $\mathfrak{sp}_n(\mathbb{R})$ and defines its \mathbb{Z} -gradation. Consider the Toda system associated with this \mathbb{Z} -gradation.

One can verify that the Lie group G_0 in the case under consideration is formed by the block $2n \times 2n$ matrices a of the form

$$a = \begin{pmatrix} b & 0 \\ 0 & (b^T)^{-1} \end{pmatrix},$$

where b is an arbitrary element of $GL_n(\mathbb{R})$. Hence, the subgroup G_0 is isomorphic to the Lie group $GL_n(\mathbb{R})$, and the mapping γ entering the general Toda equation (2.2), can be parameterized as

$$\gamma = \begin{pmatrix} \Gamma & 0 \\ 0 & (\Gamma^T)^{-1} \end{pmatrix}, \tag{7.1}$$

where the mapping Γ takes values in $GL_n(\mathbb{R})$. The general form of the mappings a_- and a_+ is again given by (2.6). Here the mappings A_- and A_+ must satisfy the relations (2.7) and

$$A_-^T = A_-, \quad A_+^T = A_+.$$

Putting $A_- = A_+ = I_n$ we come to the following Toda equations

$$\partial_+(\Gamma^{-1}\partial_-\Gamma) = -(\Gamma^T\Gamma)^{-1}. \tag{7.2}$$

In the case of $n = 1$ the mapping Γ is just an ordinary function on M taking values in \mathbb{R}^\times . If the function Γ is continuous, then it is either positive or negative. For a positive function Γ one can write $\Gamma = \exp F$ and Eq. (7.2) takes the form

$$\partial_+\partial_-\ln F = -\exp(-2F),$$

that is the well-known Liouville equation. Therefore, it is natural to call the matrix differential equation (7.2) the non-abelian Liouville equation.

The system under consideration possesses a WZNW-type symmetry

$$\Gamma \rightarrow \Lambda_+ \Gamma \Lambda_-, \tag{7.3}$$

where the matrix-valued functions Λ_- and Λ_+ satisfy the conditions

$$\partial_+\Lambda_- = 0, \quad \partial_-\Lambda_+ = 0, \quad \Lambda_-^T = \Lambda_-^{-1}, \quad \Lambda_+^T = \Lambda_+^{-1}.$$

It is also conformally invariant. Here the action of the conformal transformations on Γ is defined as

$$\Gamma(z^-, z^+) \rightarrow [\partial_-\zeta^-(z^-)\partial_+\zeta^+(z^+)]^{-1/2}\Gamma(\zeta^+(z^+), \zeta^-(z^-)). \tag{7.4}$$

The procedure described in Section 3, leads now to the following matrix characteristic integrals

$$W_1 = -\frac{1}{2}\kappa(\Sigma_- - \Sigma_-^T), \quad W_2 = -\kappa^2(\frac{1}{2}\partial_-(\Sigma_- + \Sigma_-^T) + \Sigma_- \Sigma_-^T), \tag{7.5}$$

$$\bar{W}_1 = -\frac{1}{2}\kappa(\bar{\Sigma}_+ - \bar{\Sigma}_+^T), \quad \bar{W}_2 = -\kappa^2(\frac{1}{2}\partial_-(\bar{\Sigma}_+ + \bar{\Sigma}_+^T) + \bar{\Sigma}_+ \bar{\Sigma}_+^T), \tag{7.6}$$

where

$$\Sigma_- = \Gamma^{-1} \partial_- \Gamma, \quad \bar{\Sigma}_+ = \partial_+ \Gamma \Gamma^{-1}.$$

Here we have

$$W_1^T = -W_1, \quad \bar{W}_1^T = -\bar{W}_1, \quad W_2^T = W_2, \quad \bar{W}_2^T = \bar{W}_2.$$

Therefore, in this case there are $2n^2$ independent characteristic integrals.

It is convenient to define the scalar product in $\mathfrak{sp}_n(\mathbb{R})$ as

$$B(x, y) = \frac{1}{2} \text{tr}(xy).$$

Taking into account the relations (4.10), (4.11), (7.5) and (7.6) we come to the following expressions for the non-zero components of the conformally improved energy–momentum tensor

$$T'_{--} = \frac{1}{2\kappa} \text{tr}[W_1^2 - 2W_2], \quad T'_{++} = \frac{1}{2\kappa} \text{tr}[\bar{W}_1^2 - 2\bar{W}_2]. \tag{7.7}$$

Proceed now to the Hamiltonian formalism. It is clear that the matrices

$$E_i^j = \begin{pmatrix} e_i^j & 0 \\ 0 & -(e_i^j)^T \end{pmatrix}, \quad i, j = 1, \dots, n,$$

form a basis of the Lie algebra \mathfrak{g}_0 and one has

$$B(E_i^j, E_k^l) = \delta_i^l \delta_k^j.$$

Using the equality

$$e_i^j \otimes e_j^i = (e_i^j)^T \otimes (e_j^i)^T, \tag{7.8}$$

we see that the element $C \in \mathfrak{g}_0 \otimes \mathfrak{g}_0$ is again given by the formula (6.1). Let us use for the mapping γ the parameterization (7.1) and for the quantities j and \bar{j} the parameterizations

$$j = \begin{pmatrix} \mathcal{J} & 0 \\ 0 & -\mathcal{J}^T \end{pmatrix}, \quad \bar{j} = \begin{pmatrix} \bar{\mathcal{J}} & 0 \\ 0 & -\bar{\mathcal{J}}^T \end{pmatrix},$$

where the functions \mathcal{J} and $\bar{\mathcal{J}}$ take values in $\mathfrak{gl}_n(\mathbb{R})$. Now the relations (5.7)–(5.9) give

$$\begin{aligned} \{\mathcal{J}(x) \otimes \mathcal{J}(x')\} &= -[C_n, I_n \otimes \mathcal{J}(x)] \delta(x - x') - 2\kappa C_n \delta'(x - x'), \\ \{\bar{\mathcal{J}}(x) \otimes \bar{\mathcal{J}}(x')\} &= [C_n, I_n \otimes \bar{\mathcal{J}}(x)] \delta(x - x') + 2\kappa C_n \delta'(x - x'), \\ \{\mathcal{J}(x) \otimes \bar{\mathcal{J}}(x')\} &= 0, \end{aligned}$$

and the relations (5.10) and (5.11) imply

$$\begin{aligned} \{\Gamma(x) \otimes \mathcal{J}(x')\} &= -(\Gamma(x) \otimes I_n) C_n \delta(x - x'), \\ \{\Gamma(x) \otimes \bar{\mathcal{J}}(x')\} &= -C_n (\Gamma(x) \otimes I_n) \delta(x - x'). \end{aligned}$$

The Hamiltonian counterparts of the characteristic integrals W_1 and W_2 are

$$\mathcal{W}_1 = \mathcal{J} - \mathcal{J}^T, \tag{7.9}$$

$$\mathcal{W}_2 = -\mathcal{J}\mathcal{J}^T - \kappa(\partial_x \mathcal{J} + \partial_x \mathcal{J}^T) + \kappa^2 \Gamma^{-1}(\Gamma^T)^{-1}. \tag{7.10}$$

To find the Poisson brackets for the characteristic integrals \mathcal{W}_1 and \mathcal{W}_2 we need to know the Poisson brackets between $\Gamma(x)$, $\mathcal{J}(x)$ and $\Gamma^T(x)$, $\mathcal{J}^T(x)$. They can be found in the following way. Let σ be a linear operator on $\text{Mat}_n(\mathbb{R})$ acting as

$$\sigma(a) = a^T.$$

From (D.1) it follows that

$$\{\mathcal{J}(x) \otimes \mathcal{J}^T(x')\} = (\text{id}_{\text{Mat}_n(\mathbb{R})} \otimes \sigma)(\{\mathcal{J}(x) \otimes \mathcal{J}(x')\}),$$

hence, we have

$$\{\mathcal{J}(x) \otimes \mathcal{J}^T(x')\} = [\tilde{C}_n, I_n \otimes \mathcal{J}^T(x)] \delta(x - x') - 2\kappa \tilde{C}_n \delta'(x - x'),$$

where

$$\tilde{C}_n = (\text{id}_{\text{Mat}_n(\mathbb{R})} \otimes \sigma)(C_n) = e_i^j \otimes (e_j^i)^T.$$

Note that the element \tilde{C}_n can also be defined as

$$\tilde{C}_n = (\sigma \otimes \text{id}_{\text{Mat}_n(\mathbb{R})})(C_n) = (e_j^i)^T \otimes e_j^i.$$

Using this relation and the equality (7.8), we obtain

$$\begin{aligned} \{\mathcal{J}^T(x) \otimes \mathcal{J}(x')\} &= -[\tilde{C}_n, I_n \otimes \mathcal{J}(x)] \delta(x - x') - 2\kappa \tilde{C}_n \delta'(x - x'), \\ \{\mathcal{J}^T(x) \otimes \mathcal{J}^T(x')\} &= [C_n, I_n \otimes \mathcal{J}^T(x)] \delta(x - x') - 2\kappa C_n \delta'(x - x'). \end{aligned}$$

In a similar way we come to the expressions

$$\begin{aligned} \{\Gamma(x) \otimes \mathcal{J}^T(x')\} &= -(\Gamma(x) \otimes I_n) \tilde{C}_n \delta(x - x'), \\ \{\Gamma^T(x) \otimes \mathcal{J}(x')\} &= -\tilde{C}_n (\Gamma^T(x) \otimes I_n) \delta(x - x'), \\ \{\Gamma^T(x) \otimes \mathcal{J}^T(x')\} &= -C_n (\Gamma^T(x) \otimes I_n) \delta(x - x'). \end{aligned}$$

It can be verified that the element \tilde{C}_n satisfies the relations

$$\begin{aligned} \tilde{C}_n(a \otimes b) &= \tilde{C}_n(I_n \otimes a^T b) = \tilde{C}_n(b^T a \otimes I_n), \\ (a \otimes b) \tilde{C}_n &= (I_n \otimes b a^T) \tilde{C}_n = (a b^T \otimes I_n) \tilde{C}_n \end{aligned}$$

which are used in obtaining the Poisson brackets for \mathcal{W}_1 and \mathcal{W}_2 . These Poisson brackets have the form

$$\begin{aligned} \{\mathcal{W}_1(x) \otimes \mathcal{W}_1(x')\} &= -[C_n - \tilde{C}_n, I_n \otimes \mathcal{W}_1(x)] \delta(x - x') - 4\kappa(C_n - \tilde{C}_n) \delta'(x - x'), \\ \{\mathcal{W}_1(x) \otimes \mathcal{W}_2(x')\} &= -[C_n - \tilde{C}_n, I_n \otimes \mathcal{W}_2(x)] \delta(x - x') \\ &\quad - \kappa[C_n - \tilde{C}_n, I_n \otimes \mathcal{W}_1(x')]_+ \delta'(x - x'), \\ \{\mathcal{W}_2(x) \otimes \mathcal{W}_2(x')\} &= (I_n \otimes \mathcal{W}_2(x))(C_n + \tilde{C}_n)(I_n \otimes \mathcal{W}_1(x)) \delta(x - x') \\ &\quad - (I_n \otimes \mathcal{W}_1(x))(C_n + \tilde{C}_n)(I_n \otimes \mathcal{W}_2(x)) \delta(x - x') \\ &\quad - \frac{1}{2}\kappa^2[C_n + \tilde{C}_n, I_n \otimes \partial_x^2 \mathcal{W}_1(x)] \delta(x - x') \\ &\quad + \kappa[C_n + \tilde{C}_n, I_n \otimes (\mathcal{W}_2(x) + \mathcal{W}_2(x'))]_+ \delta'(x - x') \\ &\quad - \kappa(I_n \otimes \mathcal{W}_1(x))(C_n + \tilde{C}_n)(I_n \otimes \mathcal{W}_1(x)) \delta'(x - x') \\ &\quad - \kappa(I_n \otimes \mathcal{W}_1(x'))(C_n + \tilde{C}_n)(I_n \otimes \mathcal{W}_1(x')) \delta'(x - x') \\ &\quad + \frac{3}{2}\kappa^2[C_n + \tilde{C}_n, I_n \otimes (\mathcal{W}_1(x) + \mathcal{W}_1(x'))] \delta''(x - x') \\ &\quad + 4\kappa^3(C_n + \tilde{C}_n) \delta'''(x - x'). \end{aligned}$$

Note that the first two expressions above can be obtained from the relations (6.9) and (6.10) replacing C_n by $C_n - \tilde{C}_n$. One can also obtain the third expression from (6.11) replacing C_n by $C_n + \tilde{C}_n$. The same procedure can be used to obtain the Poisson brackets for the Hamiltonian counterparts of the characteristic integrals $\bar{\mathcal{W}}_1$ and $\bar{\mathcal{W}}_2$, which are of the form

$$\bar{\mathcal{W}}_1 = \bar{\mathcal{J}} - \bar{\mathcal{J}}^T, \tag{7.11}$$

$$\bar{\mathcal{W}}_2 = -\bar{\mathcal{J}}^T \bar{\mathcal{J}} + \kappa(\partial_x \bar{\mathcal{J}} + \partial_x \bar{\mathcal{J}}^T) + \kappa^2(\Gamma^T)^{-1} \Gamma^{-1}, \tag{7.12}$$

from the relations (6.14)–(6.16).

As follows from (7.7) the Hamiltonian counterparts of the non-vanishing components of the energy–momentum tensor are

$$\mathcal{T}'_{--} = \frac{1}{2\kappa} \text{tr}[\mathcal{W}_1^2 - 2\mathcal{W}_2], \quad \mathcal{T}'_{++} = \frac{1}{2\kappa} \text{tr}[\bar{\mathcal{W}}_1^2 - 2\bar{\mathcal{W}}_2]. \tag{7.13}$$

The quantities $\mathcal{V}(x)$ and $\bar{\mathcal{V}}(x)$, defined by the relation (6.23), again give two copies of the Virasoro algebra:

$$\{\mathcal{V}(x), \mathcal{V}(x')\} = -(\mathcal{V}(x) + \mathcal{V}(x')) \delta'(x - x') + \frac{1}{2}\kappa n \delta'''(x - x'), \tag{7.14}$$

$$\{\bar{\mathcal{V}}(x), \bar{\mathcal{V}}(x')\} = (\bar{\mathcal{V}}(x) + \bar{\mathcal{V}}(x')) \delta'(x - x') - \frac{1}{2}\kappa n \delta'''(x - x'), \tag{7.15}$$

$$\{\mathcal{V}(x), \bar{\mathcal{V}}(x')\} = 0. \tag{7.16}$$

They generate the conformal transformations (7.4). It can be shown that the characteristic integrals $\mathcal{W}_1(x)$ and $\bar{\mathcal{W}}_1(x)$ have the conformal weight 1 with respect to $\mathcal{V}(x)$ and $\bar{\mathcal{V}}(x)$

respectively. The conformal weight of the characteristic integrals $\mathcal{W}_2(x)$ and $\bar{\mathcal{W}}_2(x)$ with respect to $\mathcal{V}(x)$ and $\bar{\mathcal{V}}(x)$, respectively, is equal to 2.

We will not write explicit expressions for the infinitesimal symmetry transformations generated by the characteristic integrals. They are similar to the transformations described by relations (6.28), (6.29), (6.32) and (6.33). Note only that the characteristic integrals \mathcal{W}_1 and $\bar{\mathcal{W}}_1$ generate WZNW-type symmetry transformations given by (7.3).

8. Conclusion

We found the classical W -algebras and the corresponding infinitesimal symmetry transformations for the simplest non-abelian Toda systems associated with the Lie groups $GL_{2n}(\mathbb{R})$ and $Sp_n(\mathbb{R})$. The block matrix structure of the systems under consideration results in the fact that the generators of the W -algebras appear as matrix-valued quantities. Actually, it is this fact that gives us a possibility to write the defining relations in a compact form.

To obtain the W -algebras for the case of the Toda systems related to the Lie group $Sp_n(\mathbb{R})$ one could also use the fact that the symplectic group is a subgroup of the general linear group, implementing the reduced phase space formalism. In such a case, one should work with the corresponding Dirac bracket. However, the calculations one should perform along that line of approach turn out to be more cumbersome than those we have done. So, the more direct approach to the problem presented here is certainly preferable, at least from a technical point of view.

It is worth to note that the generators of the W -algebras obtained in the paper have the conformal spin 1 or 2. Nevertheless, we gain non-linear defining relations. It is not usual for the theory of W -algebras, although it was observed in the theory of V -algebras. The latter are also extensions of the Virasoro algebra, but they allow for non-local terms in expressions for the Poisson brackets of generators [37–41]. It seems that they can be obtained from the W -algebras for non-abelian Toda systems by imposing the constraints saying that the generators of the WZNW-type symmetry, \mathcal{W}_1 and $\bar{\mathcal{W}}_1$ in our case, are equal to zero, but the explicit relationship requires further investigation. The main problem here is to determine the structure of the reduced phase space in the case when the group formed by the WZNW-type symmetry transformations is non-abelian.

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Appendix A. Coordinates and metrics conventions

Let M be a two-dimensional orientable Riemannian manifold M with metric tensor η of index 1. Arbitrary coordinates on M are denoted by z^i and for the partial derivatives we use

the notation $\partial_i = \partial/\partial z^i$. Starting from the local representation for η ,

$$\eta = \eta_{ij} dz^i \otimes dz^j,$$

we define the quantities η^{ij} by

$$\eta^{ik} \eta_{kj} = \delta^i_j,$$

and denote $\det\|\eta_{ij}\|$ simply by η .

In the paper we deal with a flat two-dimensional manifold M and use flat Minkowski coordinates z^0, z^1 , such that the metric tensor is

$$\eta = -dz^0 \otimes dz^0 + dz^1 \otimes dz^1.$$

The light-front coordinates z^- and z^+ are introduced by the relations

$$z^- = \frac{1}{2}(z^0 - z^1), \quad z^+ = \frac{1}{2}(z^0 + z^1). \tag{A.1}$$

The inverse transformation to the coordinates z^0 and z^1 is given by

$$z^0 = z^- + z^+, \quad z^1 = -z^- + z^+.$$

Using the light-front coordinates, we obtain for the metric tensor

$$\eta = -2 dz^- \otimes dz^+ - 2 dz^+ \otimes dz^-. \tag{A.2}$$

The connection of partial derivatives is

$$\partial_- = \partial_0 - \partial_1, \quad \partial_+ = \partial_0 + \partial_1, \quad \partial_0 = \frac{1}{2}(\partial_- + \partial_+), \quad \partial_1 = \frac{1}{2}(-\partial_- + \partial_+).$$

Appendix B. Some information on matrix Lie groups

Let G be a real matrix Lie group or, in other words, a Lie subgroup of the Lie group $GL_N(\mathbb{R})$. Denote by y^μ some local coordinates on G and by g the matrix-valued function which transforms the coordinates $y^\mu(a)$ of the element $a \in G$ into the element a itself. The left-invariant Maurer–Cartan form θ can be written as

$$\theta = g^{-1}(y) dg(y),$$

where the matrix-valued function g^{-1} is defined by the equality

$$g^{-1}(y)g(y) = I_N.$$

It is easy to verify that θ satisfies the relation

$$d\theta + \theta \wedge \theta = 0. \tag{B.1}$$

Using the basis of 1-forms dy^μ , one can write

$$\theta = g^{-1}(y)\partial_\mu g(y) dy^\mu = \theta_\mu(y) dy^\mu, \tag{B.2}$$

where $\theta_\mu(y)$ are matrix-valued functions on M . Recall that the Maurer–Cartan form is a \mathfrak{g} -valued one-form and therefore the functions $\theta_\mu(y)$ take values in the Lie algebra \mathfrak{g} of the Lie group G . The relation (B.1) is equivalent to the equalities

$$\partial_\mu \theta_\nu(y) - \partial_\nu \theta_\mu(y) + [\theta_\mu(y), \theta_\nu(y)] = 0. \tag{B.3}$$

Choose some basis $\{e_\alpha\}$ of \mathfrak{g} and denote by $f_{\beta\gamma}^\alpha$ the corresponding structure constants,

$$[e_\alpha, e_\beta] = e_\gamma f_{\alpha\beta}^\gamma.$$

Expand the functions $\theta_\mu(y)$ over the basis $\{e_\alpha\}$,

$$\theta_\mu(y) = e_\alpha \theta_\mu^\alpha(y). \tag{B.4}$$

This gives the following representation for the left-invariant Maurer–Cartan form:

$$\theta = e_\alpha \theta_\mu^\alpha(y) dy^\mu. \tag{B.5}$$

The relation (B.3) implies that

$$\partial_\mu \theta_\nu^\alpha(y) - \partial_\nu \theta_\mu^\alpha(y) + f_{\beta\gamma}^\alpha \theta_\mu^\beta(y) \theta_\nu^\gamma(y) = 0. \tag{B.6}$$

Denote by $X_\alpha^\mu(y)$ the functions satisfying the relation

$$X_\alpha^\mu(y) \theta_\nu^\alpha(y) = \delta_\nu^\mu. \tag{B.7}$$

Note that the functions $X_\alpha^\mu(y)$ are the components of the left-invariant vector fields $X_\alpha = X_\alpha^\mu(y) \partial_\mu$ on G . It is not difficult to see that (B.6) implies

$$X_\alpha^\mu(y) \partial_\mu X_\beta^\nu(y) - X_\beta^\mu(y) \partial_\mu X_\alpha^\nu(y) = X_\gamma^\nu(y) f_{\alpha\beta}^\gamma \tag{B.8}$$

that, in terms of the vector fields X_α , can be written as

$$[X_\alpha, X_\beta] = X_\gamma f_{\alpha\beta}^\gamma.$$

The right-invariant Maurer–Cartan form

$$\bar{\theta} = dg(y)g^{-1}(y)$$

satisfies the relation

$$d\bar{\theta} - \bar{\theta} \wedge \bar{\theta} = 0.$$

Introducing the local expansion

$$\bar{\theta} = e_\alpha \bar{\theta}_\mu^\alpha(y) dy^\mu, \tag{B.9}$$

we obtain for the functions $\bar{\theta}_\mu^\alpha(y)$ the following equalities:

$$\partial_\mu \bar{\theta}_\nu^\alpha(y) - \partial_\nu \bar{\theta}_\mu^\alpha(y) - f_{\beta\gamma}^\alpha \bar{\theta}_\mu^\beta(y) \bar{\theta}_\nu^\gamma(y) = 0. \tag{B.10}$$

The functions $\bar{X}_\alpha^\mu(y)$ defined by

$$\bar{X}_\alpha^\mu(y) \bar{\theta}_\nu^\alpha(y) = \delta_\nu^\mu \tag{B.11}$$

are the components of the right-invariant vector fields $\bar{X}_\alpha = \bar{X}_\alpha^\mu(y)\partial_\mu$ on G . The equalities (B.10) imply that

$$\bar{X}_\alpha^\mu(y)\partial_\mu \bar{X}_\beta^\nu(y) - \bar{X}_\beta^\mu(y)\partial_\mu \bar{X}_\alpha^\nu(y) = -\bar{X}_\gamma^\nu(y)f_{\alpha\beta}^\gamma \tag{B.12}$$

that is equivalent to

$$[\bar{X}_\alpha, \bar{X}_\beta] = -\bar{X}_\gamma f_{\alpha\beta}^\gamma.$$

From (B.2) and (B.4) we obtain

$$X_\alpha g(y) = g(y)e_\alpha. \tag{B.13}$$

This means that the vector fields X_α are generators of right shifts in the Lie group G . Correspondingly, the equality

$$\bar{X}_\alpha g(y) = e_\alpha g(y) \tag{B.14}$$

tells us that the vector fields \bar{X}_α are generators of left shifts. Now, using (B.13) and (B.14), we see that

$$[X_\alpha, \bar{X}_\beta]g(y) = 0$$

that implies the equality

$$[X_\alpha, \bar{X}_\beta] = 0.$$

In terms of the components we present the above equality in the form

$$X_\alpha^\mu(y)\partial_\mu \bar{X}_\beta^\nu(y) - \bar{X}_\beta^\mu(y)\partial_\mu X_\alpha^\nu(y) = 0.$$

Since we are working with a matrix Lie group G , the adjoint representation of G can be defined by the relation

$$\text{Ad}(a)x = axa^{-1},$$

and the matrix of $\text{Ad}(a)$ with respect to the basis $\{e_\alpha\}$ of \mathfrak{g} is defined by the equality

$$\text{Ad}(a)e_\alpha = e_\beta \text{Ad}_\alpha^\beta(a).$$

Consider the action of the left-invariant vector field X_α on the matrix-valued function $ge_\beta g^{-1}$. Using the equality

$$X_\alpha g^{-1}(y) = -g^{-1}(y)X_\alpha g(y)g^{-1}(y)$$

and the relation (B.13), we obtain

$$X_\alpha(g(y)e_\beta g^{-1}(y)) = g(y)e_\gamma g^{-1}(y)f_{\alpha\beta}^\gamma.$$

In terms of the matrix elements we obtain

$$X_\alpha(\text{Ad}_\beta^\gamma(g(y))) = \text{Ad}_\delta^\gamma(g(y))f_{\alpha\beta}^\delta.$$

It is not difficult to show that

$$X_\alpha(\text{Ad}_\beta^\gamma(g^{-1}(y))) = -f_{\alpha\delta}^\gamma \text{Ad}_\beta^\delta(g^{-1}(y)). \tag{B.15}$$

Recall that for any $a \in G$ the operator $\text{Ad}(a)$ is an automorphism of \mathfrak{g} :

$$\text{Ad}(a)[x, y] = [\text{Ad}(a)x, \text{Ad}(a)y].$$

In terms of components this equality takes the form

$$\text{Ad}_\delta^\alpha(a) f_{\beta\gamma}^\delta = f_{\varepsilon\zeta}^\alpha \text{Ad}_\beta^\varepsilon(a) \text{Ad}_\gamma^\zeta(a).$$

The right-invariant Maurer–Cartan form and the left-invariant one are connected by the relation

$$\bar{\theta} = g(y)\theta g^{-1}(y).$$

Using (B.5) and (B.9), we obtain the equality

$$\text{Ad}_\beta^\alpha(g(y)) = \bar{\theta}_\mu^\alpha(y) X_\beta^\mu(y). \tag{B.16}$$

Suppose that the Lie algebra \mathfrak{g} is endowed with a non-degenerate symmetric invariant scalar product. This means that there is given a bilinear mapping $B : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$, satisfying the relations

$$B(x, y) = B(y, x), \tag{B.17}$$

$$B([x, y], z) = B(x, [y, z]), \tag{B.18}$$

and the condition that if $B(x, y) = 0$ for all $y \in \mathfrak{g}$ then $x = 0$.

From (B.17) it follows that for the quantities

$$c_{\alpha\beta} = B(e_\alpha, e_\beta) \tag{B.19}$$

one has

$$c_{\alpha\beta} = c_{\beta\alpha},$$

and the non-degeneracy of the scalar product B implies that the matrix $\|c_{\alpha\beta}\|$ is invertible. Using the relation (B.18), one can show that the quantities

$$f_{\alpha\beta\gamma} = c_{\alpha\delta} f_{\beta\gamma}^\delta$$

are totally antisymmetric with respect to the indices α, β and γ . It can also be shown that in the case under consideration

$$f_{\beta\alpha}^\alpha = 0.$$

This equality implies that if the Lie group G is connected then it is unimodular.

Actually we assume in the paper that the scalar product B is Ad-invariant. It means that for any $a \in G$ and any $x, y \in \mathfrak{g}$ one has

$$B(\text{Ad}(a)x, \text{Ad}(a)y) = B(x, y).$$

In terms of components and matrix elements we have

$$c_{\gamma\delta} \text{Ad}_\alpha^\gamma(a) \text{Ad}_\beta^\delta(a) = c_{\alpha\beta}.$$

Using the equality (B.16), we obtain

$$c_{\alpha\beta} \bar{\theta}_\mu^\alpha(y) \bar{\theta}_\nu^\beta(y) = c_{\alpha\beta} \theta_\mu^\alpha(y) \theta_\nu^\beta(y). \tag{B.20}$$

The last expressions establish the bi-invariant metric tensor on the G -group manifold, related to the local coordinates y^μ .

In construction of the action of the WZNW model one uses the three-form

$$\Theta = \frac{1}{3!} B(\theta_\mu(y), [\theta_\nu(y), \theta_\rho(y)]) dy^\mu \wedge dy^\nu \wedge dy^\rho. \tag{B.21}$$

Using (B.3), one can show that this form is closed, but, in general, it is not exact. Locally for some two-form

$$\lambda = \frac{1}{2!} \lambda_{\mu\nu}(y) dy^\mu \wedge dy^\nu.$$

We can write

$$\Theta = d\lambda. \tag{B.22}$$

Taking into account (B.4), we have

$$\Theta = \frac{1}{3!} f_{\alpha\beta\gamma} \theta_\mu^\alpha(y) \theta_\nu^\beta(y) \theta_\rho^\gamma(y) dy^\mu \wedge dy^\nu \wedge dy^\rho,$$

and the relation (B.22) implies

$$\partial_\mu \lambda_{\nu\rho}(y) + \partial_\nu \lambda_{\rho\mu}(y) + \partial_\rho \lambda_{\mu\nu}(y) = f_{\alpha\beta\gamma} \theta_\mu^\alpha(y) \theta_\nu^\beta(y) \theta_\rho^\gamma(y). \tag{B.23}$$

Appendix C. Current algebra

To find the Poisson brackets for $j_\alpha(x)$ write

$$\begin{aligned} & \{\pi_\mu(x) + \kappa \lambda_{\mu\rho}(\xi(x)) \partial_x \xi^\rho(x), \pi_\nu(x') + \kappa \lambda_{\nu\sigma}(\xi(x')) \partial_x \xi^\sigma(x')\} \\ & = -\kappa [\partial_\mu \lambda_{\nu\rho}(\xi(x)) + \partial_\nu \lambda_{\rho\mu}(\xi(x)) + \partial_\rho \lambda_{\mu\nu}(\xi(x))] \partial_x \xi^\rho(x) \delta(x - x'). \end{aligned}$$

Taking into account equality (B.23), we obtain

$$\begin{aligned} & \{\pi_\mu(x) + \kappa \lambda_{\mu\rho}(\xi(x)) \partial_x \xi^\rho(x), \pi_\nu(x') + \kappa \lambda_{\nu\sigma}(\xi(x')) \partial_x \xi^\sigma(x')\} \\ & = -\kappa f_{\alpha\beta\gamma} \theta_\mu^\alpha(\xi(x)) \theta_\nu^\beta(\xi(x)) \theta_\rho^\gamma(\xi(x)) \partial_x \xi^\rho(x) \delta(x - x'). \end{aligned}$$

Hence, using the equalities (B.8) we find

$$\begin{aligned} & \{-X_\alpha^\mu(\xi(x)) [\pi_\mu(x) + \kappa \lambda_{\mu\rho}(\xi(x)) \partial_x \xi^\rho(x)], \\ & -X_\beta^\nu(\xi(x')) [\pi_\nu(x') + \kappa \lambda_{\nu\sigma}(\xi(x')) \partial_x \xi^\sigma(x')]\} \\ & = -X_\gamma^\rho(\xi(x)) [\pi_\rho(x) + \kappa \lambda_{\rho\sigma}(\xi(x)) \partial_x \xi^\sigma(x)] f_{\alpha\beta}^\gamma \delta(x - x') \\ & \quad - \kappa c_{\gamma\delta} \theta_\sigma^\delta(\xi(x)) \partial_x \xi^\sigma(x) f_{\alpha\beta}^\gamma \delta(x - x'). \end{aligned} \tag{C.1}$$

It is easy to get convinced that

$$\begin{aligned} & \{-X^\mu_\alpha(\xi(x))[\pi_\mu(x) + \kappa\lambda_{\mu\rho}(\xi(x))\partial_x\xi^\rho(x)], \kappa c_{\beta\delta}\theta^\delta_\sigma(\xi(x'))\partial_x\xi^\sigma(x')\} \\ & = \kappa X^\mu_\alpha(\xi(x))c_{\beta\delta}[\partial_\mu\theta^\delta_\sigma(\xi(x)) - \partial_\sigma\theta^\delta_\mu(\xi(x))]\partial_x\xi^\sigma(x)\delta(x-x') - \kappa c_{\alpha\beta}\delta'(x-x'). \end{aligned}$$

Using the relation (B.6), we come to

$$\begin{aligned} & \{-X^\mu_\alpha(\xi(x))[\pi_\mu(x) + \kappa\lambda_{\mu\rho}(\xi(x))\partial_x\xi^\rho(x)], \kappa c_{\beta\delta}\theta^\delta_\sigma(\xi(x'))\partial_x\xi^\sigma(x')\} \\ & = \kappa c_{\gamma\delta}\theta^\delta_\sigma(\xi(x))\partial_x\xi^\sigma(x)f^\gamma_{\alpha\beta}\delta(x-x') - \kappa c_{\alpha\beta}\delta'(x-x'). \end{aligned} \tag{C.2}$$

Similarly we obtain

$$\begin{aligned} & \{\kappa c_{\alpha\gamma}\theta^\gamma_\rho(\xi(x))\partial_x\xi^\rho(x), -X^\nu_\beta(\xi(x'))[\pi_\nu(x') + \kappa\lambda_{\nu\sigma}(\xi(x'))\partial_x\xi^\sigma(x')]\} \\ & = \kappa c_{\gamma\delta}\theta^\delta_\sigma(\xi(x))\partial_x\xi^\sigma(x)f^\gamma_{\alpha\beta}\delta(x-x') - \kappa c_{\alpha\beta}\delta'(x-x'). \end{aligned} \tag{C.3}$$

Finally, the equalities (C.1)–(C.3) give the relation (5.2).

To find the expression for the Poisson brackets of \bar{j}_α we write the equality

$$\bar{j}_\alpha = -(X^\mu_\beta(\xi)[\pi_\mu + \kappa\lambda_{\mu\rho}(\xi)\partial_x\xi^\rho] + \kappa c_{\beta\gamma}\theta^\gamma_\mu(\xi)\partial_x\xi^\mu) \text{Ad}^\beta_\alpha(g^{-1}) \tag{C.4}$$

which follows from (B.16) and (B.20). Now using (C.1)–(C.3) and (B.15) we obtain (5.3). In a similar way we arrive at the relation (5.4).

Appendix D. Algebra-valued functions on a phase space

Let A_1 and A_2 be two unital algebras with the units 1_{A_1} and 1_{A_2} . The Poisson bracket of an A_1 -valued function F and an A_2 -valued function G on a symplectic manifold M is defined in the following way. Choose some basis $\{e_\alpha\}$ of A_1 and some basis $\{f_i\}$ of A_2 . Expand the functions F and G over the bases,

$$F = e_\alpha F^\alpha, \quad G = f_i G^i.$$

Then the Poisson bracket of F and G is defined as an $A_1 \otimes A_2$ -valued function

$$\{F \otimes G\} = e_\alpha \otimes f_i \{F^\alpha, G^i\}.$$

The Poisson bracket $\{F \otimes G\}$ does not depend on the choice of the bases $\{e_\alpha\}$ and $\{f_i\}$.

Introducing the linear mapping P from $A_1 \otimes A_2$ to $A_2 \otimes A_1$ defined by the relation

$$P(a \otimes b) = b \otimes a,$$

we can reformulate the usual properties of the Poisson bracket as follows:

$$\begin{aligned} \{F \otimes G\} &= -P \circ \{G \otimes F\}, & \{F \otimes GH\} &= \{F \otimes G\}(1_{A_1} \otimes H) + (1_{A_1} \otimes G)\{F \otimes H\}, \\ \{FG \otimes H\} &= (F \otimes 1_{A_2})\{G \otimes H\} + \{F \otimes H\}(G \otimes 1_{A_2}). \end{aligned}$$

It is clear that P is an isomorphism of the algebras $A_1 \otimes A_2$ and $A_2 \otimes A_1$. The Jacobi identity for the usual Poisson bracket implies

$$P_{13} \circ \{F \otimes \{G \otimes H\}\} + P_{23} \circ \{H \otimes \{F \otimes G\}\} + P_{12} \circ \{G \otimes \{H \otimes F\}\} = 0.$$

Here P_{13} is the linear mapping from $A_1 \otimes A_2 \otimes A_3$ to $A_3 \otimes A_2 \otimes A_1$ permuting the first and the third factors:

$$P_{13}(a \otimes b \otimes c) = c \otimes b \otimes a.$$

The linear mappings P_{12} and P_{23} are defined analogously.

Let σ_1 be a linear mapping from an algebra A_1 to an algebra B_1 , and σ_2 be a linear mapping from an algebra A_2 to an algebra B_2 . It can be easily shown that

$$\{\sigma_1 \circ F \otimes G\} = (\sigma_1 \otimes \text{id}_{A_2})(\{F \otimes G\}), \quad \{F \otimes \sigma_2 \circ G\} = (\text{id}_{A_1} \otimes \sigma_2)(\{F \otimes G\}). \tag{D.1}$$

Appendix E. W-algebra calculations

To obtain the expression for the Poisson bracket of \mathcal{W}_1 and \mathcal{W}_2 we find first

$$\begin{aligned} & \{\mathcal{J}^{(1)}(x) + \mathcal{J}^{(2)}(x) \otimes \mathcal{J}^{(1)}(x') \mathcal{J}^{(2)}(x')\} \\ &= -[C_n, I_n \otimes \mathcal{J}^{(1)}(x) \mathcal{J}^{(2)}(x)] \delta(x - x') - 2\kappa C_n (\mathcal{J}^{(1)}(x') \otimes I_n) \delta'(x - x') \\ & \quad - 2\kappa (\mathcal{J}^{(2)}(x') \otimes I_n) C_n \delta'(x - x'), \end{aligned} \tag{E.1}$$

$$\begin{aligned} & \{\mathcal{J}^{(1)}(x) + \mathcal{J}^{(2)}(x) \otimes -\kappa(\partial_x \mathcal{J}^{(1)}(x') - \partial_x \mathcal{J}^{(2)}(x'))\} \\ &= -[C_n, I_n \otimes -\kappa(\partial_x \mathcal{J}^{(1)}(x) - \partial_x \mathcal{J}^{(2)}(x))] \delta(x - x') \\ & \quad - \kappa [C_n, I_n \otimes (\mathcal{J}^{(1)}(x') - \mathcal{J}^{(2)}(x'))] \delta'(x - x'). \end{aligned} \tag{E.2}$$

We also need the Poisson brackets of $\mathcal{J}^{(r)}$ with $\Gamma^{(r)-1}$. To find them note that relation (6.5) implies

$$\{\mathcal{J}^{(r)}(x) \otimes \Gamma^{(s)}(x')\} = (I_n \otimes \Gamma^{(r)}(x)) C_n \delta(x - x') \delta^{rs}.$$

Writing now the relation

$$\begin{aligned} \{\mathcal{J}^{(r)}(x) \otimes \Gamma^{(s)}(x') \Gamma^{(s)-1}(x')\} &= (I_n \otimes \Gamma^{(r)}(x)) C_n (I_n \otimes \Gamma^{(s)-1}(x')) \delta(x - x') \delta^{rs} \\ & \quad + (I_n \otimes \Gamma^{(s)}(x')) \{\mathcal{J}^{(r)}(x) \otimes \Gamma^{(s)-1}(x')\} = 0, \end{aligned}$$

we obtain

$$\{\mathcal{J}^{(r)}(x) \otimes \Gamma^{(s)-1}(x')\} = -C_n (I_n \otimes \Gamma^{(r)-1}(x)) \delta(x - x') \delta^{rs}.$$

Using this equality, we come to

$$\begin{aligned} & \{ \mathcal{J}^{(1)}(x) + \mathcal{J}^{(2)}(x) \otimes \kappa^2 \Gamma^{(1)-1}(x') \Gamma^{(2)}(x') \} \\ & = -[C_n, I_n \otimes \kappa^2 \Gamma^{(1)-1}(x) \Gamma^{(2)}(x)] \delta(x - x'). \end{aligned} \tag{E.3}$$

Collecting the equalities (E.1)–(E.3), we obtain the relation (6.10).

The calculation of the Poisson bracket for \mathcal{W}_2 is more complicated. The main formulas used here are

$$\begin{aligned} & \{ \mathcal{J}^{(1)}(x) \mathcal{J}^{(2)}(x) \otimes \mathcal{J}^{(1)}(x') \mathcal{J}^{(2)}(x') \} \\ & = -[C_n, (\mathcal{J}^{(1)}(x) + \mathcal{J}^{(2)}(x)) \otimes \mathcal{J}^{(1)}(x) \mathcal{J}^{(2)}(x)] \delta(x - x') \\ & \quad - 2\kappa C_n (\mathcal{J}^{(1)}(x') \otimes \mathcal{J}^{(1)}(x)) \delta'(x - x') - 2\kappa C_n (\mathcal{J}^{(2)}(x) \otimes \mathcal{J}^{(2)}(x')) \delta'(x - x'), \\ & \mathcal{J}^{(1)}(x) \mathcal{J}^{(2)}(x) \otimes -\kappa (\partial_x \mathcal{J}^{(1)}(x') - \partial_x \mathcal{J}^{(2)}(x')) \} \\ & = \kappa [C_n, I_n \otimes \mathcal{J}^{(1)}(x) \mathcal{J}^{(2)}(x)]_+ \delta'(x - x') - 2\kappa C_n (\mathcal{J}^{(2)}(x) \otimes \mathcal{J}^{(1)}(x)) \delta'(x - x') \\ & \quad + 2\kappa^2 C_n (I_n \otimes \mathcal{J}^{(1)}(x)) \delta''(x - x') - 2\kappa^2 C_n (I_n \otimes \mathcal{J}^{(2)}(x)) \delta''(x - x'), \\ & \{ \mathcal{J}^{(1)}(x) \mathcal{J}^{(2)}(x) \otimes \kappa^2 \Gamma^{(1)-1}(x') \Gamma^{(2)}(x') \} \\ & = \kappa^2 (\mathcal{J}^{(1)}(x) \otimes \Gamma^{(1)-1}(x) \Gamma^{(2)}(x)) C_n \delta(x - x') \\ & \quad - \kappa^2 C_n (\mathcal{J}^{(2)}(x) \otimes \Gamma^{(1)-1}(x) \Gamma^{(2)}(x)) \delta(x - x'), \\ & \{ -\kappa (\partial_x \mathcal{J}^{(1)}(x) - \partial_x \mathcal{J}^{(2)}(x)) \otimes -\kappa (\partial_x \mathcal{J}^{(1)}(x') - \partial_x \mathcal{J}^{(2)}(x')) \} \\ & = \kappa^2 [C_n, I_n \otimes (\partial_x \mathcal{J}^{(1)}(x) + \partial_x \mathcal{J}^{(2)}(x))] \delta'(x - x') \\ & \quad + \kappa^2 [C_n, I_n \otimes (\mathcal{J}^{(1)}(x) + \mathcal{J}^{(2)}(x))] \delta''(x - x') + 4\kappa^3 C_n \delta'''(x - x'), \\ & \{ -\kappa (\partial_x \mathcal{J}^{(1)}(x) - \partial_x \mathcal{J}^{(2)}(x)) \otimes \kappa^2 \Gamma^{(1)-1}(x') \Gamma^{(2)}(x') \} \\ & = \kappa^3 [C_n, I_n \otimes \Gamma^{(1)-1}(x') \Gamma^{(2)}(x')]_+ \delta'(x - x'). \end{aligned}$$

Using these equalities, after some rearrangement of terms we come to the relation (6.11).

The relations (6.14)–(6.16) can be proven in the same way.

Now we will show that

$$\{ \mathcal{W}_r(x) \otimes \bar{\mathcal{W}}_s(x') \} = 0. \tag{E.4}$$

For $r = s = 1$ this is a direct consequence of the equality (6.4). For the cases $r = 1, s = 2$ and $r = 2, s = 1$ we come to (E.4) through the relations

$$\begin{aligned} & \{ \mathcal{J}^{(1)}(x) + \mathcal{J}^{(2)}(x) \otimes \kappa^2 \Gamma^{(2)}(x') \Gamma^{(1)-1}(x') \} = 0, \\ & \{ \kappa^2 \Gamma^{(1)-1}(x) \Gamma^{(2)}(x) \otimes \bar{\mathcal{J}}^{(1)}(x') + \bar{\mathcal{J}}^{(2)}(x') \} = 0. \end{aligned}$$

Finally, using the equalities

$$\begin{aligned}
 & \{\mathcal{J}^{(1)}(x)\mathcal{J}^{(2)}(x)\otimes\kappa^2\Gamma^{(2)}(x')\Gamma^{(1)-1}(x')\} \\
 &= \kappa^2(\mathcal{J}^{(1)}(x)\Gamma^{(1)-1}(x)\otimes\Gamma^{(2)}(x)-\Gamma^{(1)-1}(x)\otimes\Gamma^{(2)}(x)\mathcal{J}^{(2)}(x))\delta(x-x'), \\
 & \{\kappa^2\Gamma^{(1)-1}(x)\Gamma^{(2)}(x)\otimes\bar{\mathcal{J}}^{(2)}(x')\bar{\mathcal{J}}^{(1)}(x')\} \\
 &= -\kappa^2(\Gamma^{(1)-1}(x)\bar{\mathcal{J}}^{(1)}(x)\otimes\Gamma^{(2)}(x)-\Gamma^{(1)-1}(x)\otimes\bar{\mathcal{J}}^{(2)}(x)\Gamma^{(2)}(x))\delta(x-x'), \\
 & \{-\kappa(\partial_x\mathcal{J}^{(1)}(x)-\partial_x\mathcal{J}^{(2)}(x))\otimes\kappa^2\Gamma^{(2)}(x')\Gamma^{(1)-1}(x')\} \\
 &+ \{\kappa^2\Gamma^{(1)-1}(x)\Gamma^{(2)}(x)\otimes\kappa(\partial_x\bar{\mathcal{J}}^{(1)}(x')-\partial_x\bar{\mathcal{J}}^{(2)}(x'))\} \\
 &= 2\kappa^3(-\Gamma^{(1)-1}(x)\partial_x\Gamma^{(1)}(x)\Gamma^{(1)-1}(x)\otimes\Gamma^{(2)}(x) \\
 &+ \Gamma^{(1)-1}(x)\otimes\partial_x\Gamma^{(2)}(x))\delta(x-x'),
 \end{aligned}$$

and taking into account the identity

$$\mathcal{J}^{(r)} = \Gamma^{(r)-1}\bar{\mathcal{J}}^{(r)}\Gamma^{(r)} + 2\kappa\Gamma^{(r)-1}\partial_x\Gamma^{(r)},$$

we see that (E.4) is valid for the case $r = s = 2$ as well.

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